m3pm16l24.tex
Lecture 24. 7.3.2013.

Proof, Wiener-Ikehara Theorem, additive form, continued
For $-b \leq v \leq b$, as $e^t \sigma(t) \uparrow$,

$$\sigma(u - v/\lambda) \leq e^{v/\lambda + b/\lambda} \sigma(u + b/\lambda) \leq e^{2b/\lambda} \sigma(u + b/\lambda).$$

So

$$\begin{aligned}
\liminf_{u\to\infty} \sigma(u + b/\lambda) e^{2b/\lambda} \int_{-b}^{b} K(v) dv &\geq \liminf \int_{-b}^{b} \sigma(u - v/\lambda) K(v) dv \\
&= (\lim \int_{-\infty}^{\infty} - \limsup \int_{-\infty}^{-b} - \limsup \int_{b}^{\infty})... \\
&\geq A - 2M \int_{b}^{\infty} (1/v^2) dv = A - 2M/b.
\end{aligned}$$

For λ and b large (e.g. take $\lambda = b \to \infty$), this gives

$$\liminf_{u\to\infty} \sigma(u) \geq A.$$

Combining,

$$\sigma(u) \to A \qquad (u \to \infty). \qquad //$$

For applications (to Number Theory), we pass to the multiplicative form. All that we need is that the extension to the closed half-plane is continuous (continuous functions are locally integrable). So the following is the special case we need of the multiplicative form of what we have proved:

Theorem (Wiener-Ikehara Theorem, multiplicative form). If $A(u) = 0$ $(u < 1)$, is non-decreasing and right-continuous, and the MST

$$\alpha(s) := \int_{1}^{\infty} u^{-s} dA(u)$$

converges for all $s = \sigma + it$ with $\sigma > 1$, and

$$\alpha(s) - c/(s - 1)$$

extends (can be continued analytically) to a continuous function in the closed half-plane $\sigma \geq 1$ – then

$$A(x) = cx + o(x) \qquad (x \to \infty).$$

7. Proof of PNT

We actually only need the special case of the Wiener-Ikehara Theorem for Dirichlet series:

Cor. (Wiener-Ikehara Theorem for Dirichlet series). If $a_n \geq 0$, and

$$\sigma(s) := \sum_1^\infty a_n/n^s$$

converges for $\sigma > 1$, and $\alpha(s) - c/(s-1)$ extends to a continuous function in $\sigma \geq 1$ – then

$$\sum_{n \leq x} a_n = cx + o(x).$$

This yields

Prime Number Theorem (PNT). $\pi(x) \sim x/\log x$, $\psi(x) \sim x$.

Proof. These are equivalent forms of PNT, by III.1. We apply the result above to the case $a_n = \Lambda_n$ of the von Mangoldt function. By II.6 L12 and III.3 L19,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_1^\infty \Lambda(n)/n^s = \frac{1}{s-1} - \gamma + a_a(s-1) + a_2(s-1)^2 + \ldots \qquad (*)$$

Because σ does not vanish on the 1-line (III.4), $-\zeta'/\zeta$ is well-behaved (continuous, indeed holomorohic) on the 1-line, *except* for the simple pole at 1 of residue 1. If we *subtract off the pole* by considering

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1},$$

we have good behaviour (continuity, holomorphy) *throughout* the 1-line. The Wiener-Ikehara Theorem for Dirichlet series gives $\psi(x) \sim x$, which is PNT. //