m3pm16l26.tex
Lecture 26. 12.3.2013.

9. Distribution of zeros.

In the last 5 lectures, we prove PNT with remainder, largely following Davenport's book [D].

Theorem. (i) ξ is an entire function of order 1.
(ii) For some constants A, B,

$$\xi(s) = e^{A+Bs} \prod_\rho (1 - s/\rho) e^{s/\rho},$$

where $\rho = \beta + i\gamma$ runs over the zeros of the zeta function ζ.

Proof. (i) From III.3 $(*)$ (L18),

$$|(s-1)\zeta(s)| = O(|s|^2).$$

From Stirling's formula, where the leading term is $z^z = \exp\{z \log z\}$, we see that the Γ factor in ξ is $O_\delta(\exp\{|s|^{1+\delta}\})$ (so order 1), and similarly for the $\pi^{-\frac{1}{2}s}$ term. So $\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ is of order 1.
(ii) In the definition of ξ, the zero at $s = 0$ is cancelled by the pole of Γ at 0, and the zero at $s = 1$ is cancelled by the pole of ζ. The result follows by the Hadamard factorization (Handout, Further Complex Analysis). //

Theorem (Partial fraction expansion). We have the partial fraction expansions

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho} \Big),$$

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} + \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho} \Big).$$

Proof. This follows by logarithmic differentiation of (ii) above for ξ, and then from the definition of ξ. //

Corollary 1.

$$\sum_\rho \frac{1}{1 + (T-\gamma)^2} = O(\log T).$$

Proof. Put $s = 2+iT$. The LHS in the partial fraction expansion is bounded in T, as it is the Dirichlet series for Λ. The first three terms on RHS are bounded, the fourth is $O(\log T)$ by Stirling's formula. So

$$\sum_\rho \Big(\frac{1}{2+iT-\rho} + \frac{1}{\rho} \Big) = O(\log T).$$

That is, $Re\ \sum_\rho (...) = O(\log T)$ (also for Im ..., but we do not need this). As

$Re\, 1/\rho = \beta/(\beta^2+\gamma^2) > 0$ and $Re(1/(2+iT-\rho) = (2-\beta)/((2-\beta)^2+(T-\gamma)^2) > 0$,

there is no cancellation, so each of the two sums is $O(\log T)$. The first gives

$$\sum_\rho \frac{2-\beta}{(2-\beta)^2 + (T-\gamma)^2} = O(\log T).$$

As ρ is in the critical strip, $0 < \beta < 1$, $1 < 2-\beta < 2$, so

$$\frac{2-\beta}{(2-\beta)^2 + (T-\gamma)^2} \geq \frac{1}{4 + (T-\gamma)^2},$$

giving the result but with 4 in place of 1 in the denominator. But this is equivalent to the result. //

Corollary 2. (i) The number of zeros ρ with $|T-\gamma| \leq 1$ is $O(\log T)$.
(ii) $\sum_{|T-\gamma|>1} (T-\gamma)^{-2} = O(\log T)$.

Proof. (i)

$$\frac{1}{2} \#\{\rho : |T-\rho| \leq 1\} \leq \sum_{\rho : |T-\gamma| \leq 1} \frac{1}{1+(T-\gamma)^2} = O(\log T).$$

(ii) Similarly, the other sum is also $O(\log T)$, and there the 1 in the denominator is unimportant. //

Corollary 3. The number $N(T)$ of zeros $\rho = \beta + i\gamma$ of σ with $|\beta| \leq T$ satisrfies

$$N(T) = O(T \log T).$$

Proof. This follows from Cor. 2 (i) by summation (or integration):

$$\int_1^x \log u du = x \log x - x + 1 \sim x \log x; \qquad \sum_{k=1}^n \log k \sim n \log n. \qquad //$$