m3pm16l27.tex

Lecture 27. 14.3.2013

Note. This result can be sharpened: $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$. In particular, there are infinitely many non-trivial zeros.

N. Levinson (1974) proved that at least a third of the non-trivial zeros are on the critical line, improved to at least 2/5 by J. B. Conrey (1989).

Lemma. For $t \geq 1$, $-1 \leq \sigma \leq 2$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho:|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log t).$$

Proof. Use the partial fraction expansion for $\zeta'(z)/\zeta(z)$ with $z = s$ and $z = 2 + it$ and subtract. As $\zeta'(2+it)/\zeta(2+it)$ is bounded in t (from the Dirichlet series for Λ), this gives

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log t) + \sum_\rho \Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big) = O(\log t) + \Sigma \quad \text{say.}$$

$$\Sigma := \sum_\rho \Big(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big) = \sum_{\rho:|t-\gamma|\leq 1} + \sum_{|t-\gamma|>1} = \Sigma_1 + \Sigma_2,$$

say. Now

$$\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)}.$$

As $|2-\sigma| \leq 3$, $\sigma - \rho$ and $2 + it - \rho$ both have imaginary part $t - \gamma$, and for complex numbers $|z_1 - z_2| \geq |x_1 - x_2|, |y_1 - y_2|$,

$$\Big|\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\Big| << |\gamma - t|^{-2}, \qquad (*)$$

$$\Sigma_1 = \sum_{\rho:|t-\gamma|\leq 1} \frac{1}{s-\rho} - \sum_{|t-\gamma|\leq 1} \frac{1}{2+it-\rho} = \sum_{\rho:|t-\gamma|\leq 1} \frac{1}{s-\rho} + O(\log t),$$

as by Corollary 2 there are $O(\log t)$ terms, and as $0 < \beta < 1$ (so $1 < 2 - \beta < 2$), $0 \leq |t-\gamma| \leq 1$, $1 \leq (2-\beta)^2 + (t-\gamma)^2$, so $1/|2+it-\rho| = 1/\sqrt{(2-\beta)^2 + (t-\gamma)^2} \leq 1$. By $(*)$ and Corollary 2 (ii), $\Sigma_2 = O(\log t)$. Combining, the Lemma follows. //

We shall use the next result in the proof of the Riemann-von Mangoldt formula in III.10. It says that the bound we need for ζ'/ζ holds 'all the way horizontally, close enough vertically'.

Proposition. For $T \geq 2$, there exists U, $|T - U| \leq 1$, with

$$\zeta'(s)/\zeta(s) = O(\log^2 T) \qquad \forall\ s = \sigma + iU, \quad -1 \leq \sigma \leq 2.$$

Proof. By Cor. 2(ii), there are $O(\log T)$ zeros $\rho = \beta + i\gamma$ with $|\gamma - T| \leq 1$. We can choose $U \in [T-1, T+1]$ to avoid all the γs by $>> (\log T)^{-1}$ (or this would contradict the boundedness of $[T-1, T+1]$). By the Lemma,

$$\zeta'(s)/\zeta(s) = \sum_{\rho:|\gamma - U| \leq 1} 1/(s-\rho) + O(\log T) \qquad (s = \sigma + iU).$$

There are $O(\log T)$ terms, each of magnitude $O(\log T)$, giving $O(\log^2 T)$. //

10. Perron's formula and the Riemann-von Mangoldt formula.

Theorem (Perron's formula. For $c \in (1, 2]$, $T > 2$, $x \neq 1$ positive and real, H the Heaviside function,

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} ds = H(x) + O\Big(\frac{x^c}{T|\log x|}\Big).$$

Proof. (i) Take $x < 1$, C the rectangular contour $C_1 \cup \ldots \cup C_4$, where $C_1 = [c - iT, c + iT]$, $C_2 = [c + iT, U + iT]$, $C_3 = [U + iT, U - iT]$, $C_4 = [U - iT, c - iT]$, with U large and real. Write $I_j := \int_{C_j} (x^s/s) ds$. By Cauchy's theorem, $I_1 + \ldots + I_4 = 0$, as the integrand has only the singularity at $s = 0$, and this is outside the contour. Also $I_3 \to 0$ as $U \to \infty$, as $x < 1$. On C_2, $x^s/s = x^{iT}.x^\sigma/s$, and $|s| \geq T$, so

$$|I_2| \leq \int_c^U x^\sigma d\sigma/T \leq \int_c^\infty \ldots = O\Big(\frac{x^c}{T \log x}\Big),$$

and similarly for I_4. Letting $U \to \infty$ gives the result.
(ii) If $x > 1$, take C the rectangle with vertices $-U \pm iT$, $c \pm iT$. The pole of $x^s/s = e^{s \log x}/s$ at 0 (of residue 1) is now inside C; use Cauchy's residue theorem. //