m3pm16l28.tex
Lecture 28. 19.3.2013

We call x a *half-integer* if it has the form $x = n + \frac{1}{2}$.

Lemma 1. For $x > 1$ a half-integer,

$$\sum_{\frac{1}{2}x<n<\frac{3}{2}x} |\log(n/x)|^{-1} = O(x \log x).$$

Proof. In the summation, $|\log(n/x)| >> |1 - n/x| = |x - n|/x$. So

$$\sum_{\frac{1}{2}x<n<\frac{3}{2}x} |\log(n/x)|^{-1} << x \sum_{\frac{1}{2}x<n<\frac{3}{2}x} |x-n|^{-1} << x \sum_{n \le x} 1/n = O(x \log x). \quad //$$

Lemma 2. If $x > 1$ is a half-integer, $2 \le T \le x$ and $c := 1 + 1/\log x$,

$$\psi(x) := \sum_{n \le x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} . \frac{x^s}{s} ds + O\Big(\frac{x \log^2 x}{T}\Big).$$

Proof. Apply Perron's formula termwise to $-\zeta'(s)/\zeta(s) = \sum_1^\infty \Lambda(n)/n^s$:

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} . \frac{x^s}{s} ds = \sum_1^\infty \Lambda(n) . \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{(x/n)^s}{s} ds$$

$$= \sum_{1 \le n \le x} \Lambda(n) + O(\frac{1}{T} \sum_1^\infty \Lambda(n)(x/n)^c (\log|x/n|)^{-1}) = \sum_{n \le x} \Lambda(n) + E,$$

say. As $c = 1 + 1/\log x = 1 + o(1)$, $x^c = O(x)$. So

$$E << (x/T) \sum_1^\infty \Lambda(n) n^{-c} |\log(x/n)|^{-1}. \qquad (E)$$

(i) For $n \notin (\frac{1}{2}x, \frac{3}{2}x)$, the log term is bounded, so these terms contribute

$$<< \sum_1^\infty \Lambda(n)/n^c = -\zeta'(c)/\zeta(c).$$

As $c := 1 + 1/\log x$, $\log x = 1/(c-1)$; as $-\zeta'/\zeta$ has a simple pole at 1 of residue 1, $-\zeta'(s)/\zeta(s) \sim 1/(s-1)$ near 1 (III.3 L19), so the RHS above is:

$O(1/(c-1)) = O(\log x)$.
(ii) When $n \in (\frac{1}{2}x, \frac{3}{2}x)$, use $\Lambda(n) << \log n << \log x$ and $n^c = \exp\{c \log n\} = \exp\{(1 + 1/\log x) \log n\} \sim \exp\{(1 + 1/\log x) \log x\} \sim ex$: $\Lambda(n)/n^c << \log x/x$. By Lemma 1, this sum contributes $O(x \log x . \log x/x) = O(\log^2 x)$.

So case (ii) dominates, and (E) gives $E = O((x/T)\log^2 x)$. //

The following result is known as the Riemann-von Mangoldt formula (RvM) or the approximate explicit formula (AEF).

Theorem (Riemann-von Mangoldt formula). For $2 \leq T \leq x$ and x a half-integer,

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = x - \sum_{|\rho| \leq T} x^\rho/\rho + O\Big(\frac{x \log^2 x}{T}\Big). \qquad (RvM)$$

Proof. Changing between T and U in the Proposition of III.9 does not affect the error term. By Cor. 2 of III.9, the corresponding change in the sum is $O(x \log T/T)$, which can be absorbed into the error term. So we can assume the conclusion of the Proposition, and write T for U for convenience:

$$\zeta'(s)/\zeta(s) = O(\log^2 T) \qquad (s = \sigma + iT,\ -1 \leq \sigma \leq 2). \qquad (*)$$

With $c := 1 + 1/\log x$ as before, we evaluate

$$I_1 := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)}.\frac{x^s}{s} ds$$

by integrating round the contour $C = C_1 \cup \ldots \cup C_4$, where $C_1 = [c-iT, c+iT]$, $C_2 = [c+iT, -1+iT]$, $C_3 = [-1+iT, -1-iT]$, $C_4 = [-1-iT, c-iT]$. Write $I := \int_C$, $I_j := \int_{C_j}$. Inside C, the integrand above has:
(i) a pole at $s = 1$, of residue x ($-\zeta'/\zeta$ has a simple pole of residue 1);
(NB: this x is the *leading term*, in PNT in the form $\psi(x) \sim x$; everything else goes into the *remainder term*);
(ii) a pole at $s = 0$ (of residue $\zeta'(0)/\zeta(0)$, constant, $O(1)$);
(iii) poles at the non-trivial zeros ρ with $|\rho| < T$, of residue $-x^\rho/\rho$, by the partial fraction expansion of III.9.
So by Cauchy's residue theorem,

$$I = I_1 + \ldots + I_4 = x + O(1) - \sum_{|\rho| < T} \frac{x^\rho}{\rho}. \qquad (**)$$