m3pm16l29.tex

Lecture 29. 19.3.2013.

11. The zero-free region.

We give the classical zero-free region of Hadamard and de la Vallée Poussin. We follow Titchmarsh [T], Th. 3.8.

Theorem. For some absolute constant $c > 0$, $\zeta(s)$ has no zeros in the region

$$\sigma \geq 1 - \frac{c}{\log t} \qquad (t \geq t_0). \qquad (ZFR)$$

Proof. For $\sigma > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}}, \qquad -Re\,\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{m\sigma}} \cos(mt \log p).$$

So as in III.4, for $\sigma > 1$ and γ real (w.l.o.g. ≥ 2),

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} + 4Re\frac{\zeta'(\sigma + i\gamma)}{\zeta(\sigma + i\gamma)} - Re\frac{\zeta'(\sigma + 2i\gamma)}{\zeta(\sigma + 2i\gamma)}$$

$$= \sum_{p,m} \frac{\log p}{p^{m\sigma}} \{3 + 4\cos(m\gamma \log p) + \cos(2m\gamma \log p)\} \geq 0,$$

as $\{...\} \geq 0$ by III.4. As ζ has a simple pole at 1 of residue 1,

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1)} + O(1).$$

By the partial fraction expansion $(*)$ and Stirling's formula,

$$-\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{1}{s - \rho} + \frac{1}{\rho}\Big).$$

So ($s = \sigma + it$, $\rho = \beta + i\gamma$)

$$-Re\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_{\rho} \Big(\frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} + \frac{\beta}{\beta^2 + \gamma^2}\Big).$$

Each term in the last sum is positive (as $\frac{1}{2} \leq \beta < 1$, $\sigma > 1$). So

$$-Re\frac{\zeta'(s)}{\zeta(s)} < O(\log t).$$

Also, taking $s = \sigma + i\gamma$ with $\rho = \beta + i\gamma$ gives

$$-Re\frac{\zeta'(\sigma + i\gamma)}{\zeta(\sigma + i\gamma)} < O(\log\gamma) - \frac{1}{\sigma - \beta},$$

discarding every term (as above) except $1/(s - \rho)$.

Combining,

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + O(\log\gamma) \geq 0$$

(as $\gamma \to \infty$). So for each t_0 we can find $C > 0$ so large that

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} \geq -C\log\gamma.$$

Solving for β, this says

$$1 - \beta \geq \frac{1 - (\sigma - 1)C\log\gamma}{3/(\sigma - 1) + C\log\gamma}.$$

Here $\sigma > 1$ is free. Choose $\sigma - 1 = \frac{1}{2}/(C\log\gamma)$:

$$1 - \beta \geq \frac{\frac{1}{2}}{3C\log\gamma/\frac{1}{2} + C\log\gamma} = \frac{c}{\log\gamma}, \qquad c := 14C. \qquad //$$

2. Error terms and zero-free regions of ζ.

Landau (Handbuch, §42) shows that from de la Vallée Poussin's 1896 zero-free region $\sigma \geq 1 - a/\log t$ ($t \geq t_0$) follows $\pi(s) - li(x) = O(x\exp\{-\alpha\sqrt{\log x}\})$, for all $\alpha < \sqrt{a}$. In the other direction, Pál TURÁN (1910-76) (1950; book of 1984) showed that an error term $O(x\exp(-a(\log x)^b))$ implies a zero-free region $\sigma \geq 1 - c(\log(2 + |t|))^{(b-1)/b}$.

Taking $b = 2/3$, $c = 1/3$ corresponds to the best results known (I. M. VINOGRADOV (1891-1983) in 1958, N. M. KOROBOV in 1958):

$$\psi(x) - x = O(x\exp\{-C(\log x)^{3/5}/(\log\log x)^{1/5}\} \quad (C > 0),$$

$$\sigma \geq 1 - \frac{C}{(\log t)^{2/3}(\log\log t)^{1/3}} \qquad (t \geq t_0).$$