m3pm16l30.tex
Lecture 30. 21.3.2013.
12. PNT with Remainder.

Theorem (PNT with Remainder: Hadamard, de la Vallée Poussin). For some absolute constant $c > 0$,

$$\psi(x) = x + O(x \exp\{-c\sqrt{\log x}\}).$$

Proof. We use (RvM) (III.10) and (ZFR) (III.11). For $|\rho| \leq T$, we have $\beta \leq 1 - c/\log T$. So

$$\begin{aligned} |x^\rho| = |e^{(\beta+i\gamma)\log x}| &= e^{\beta \log x} \\ &\leq \exp\{(1 - c/\log T)\log x\} \\ &= x \exp\{-c \log x/\log T\}. \end{aligned}$$

As $\zeta(0) = -\frac{1}{2}$, the zeros of ζ are bounded away from 0, so $1/\rho$ is bounded. So

$$|x^\rho/\rho| = O(x \exp\{-c \log x/\log T\}).$$

From Cor. 3 of III.9, $N(T) = O(T \log T)$. So

$$|\sum_{|\rho|\leq T} x^\rho/\rho| = O(x.T\log T. \exp\{-c\log x/\log T\}).$$

Take

$$T = \exp\{\sqrt{(1/2)c\log x}\}.$$

Then

$$T\log T = \sqrt{(1/2)c\log x}. \exp\{\sqrt{(1/2)c\log x}\},$$

$$\exp\{-c\log x/\log T\} = \exp\{-c\log x/\sqrt{(1/2)c\log x}\} = \exp\{-\sqrt{2c\log x}\}.$$

Combining, the error term is

$$<< x.\sqrt{(1/2)c\log x}. \exp\{\sqrt{(1/2)c\log x}\}. \exp\{-\sqrt{2c\log x}\}.$$

The last two terms are $<< \exp\{-c_1 \log x\}$ for some $c_1 > 0$. As $\exp\{a\sqrt{\log x}\}$ increases (much!) faster than any power of $\log x$, the RHS is

$$<< x. \exp\{-c_2\sqrt{\log x}\}$$

for some constant $c_2 > 0$. Similarly, the other error term in (RvM), $O(x\log^2 x/T) << x\log^2 x \exp\{-\sqrt{(1/2)c\log x}\}$ is also of this form. Combining, so is $\psi(x) - x$.
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Note. 1. From the Taylor expansion for exp, for $\alpha \in (0,1)$ $\exp\{(\log x)^\alpha\}$ grows (much!) faster than any power of log. Also, for any $\lambda > 0$, $\exp\{(\log(\lambda x))^\alpha\} = \exp\{(\log\lambda + \log x)^\alpha\} \sim \exp\{(\log x)^\alpha\}$ (check) (functions with this property are called *slowly varying*). By contrast, $\exp\{(\log x)^\alpha\}$ grows (much!) more slowly than $\exp\{\log x\} = x$. These slowly varying functions provide a convenient *scale of growth*, by which we can judge the comparative precision of different forms of PNT with remainder.
2. The classical remainder (H-dlVP) of 1896 has $\alpha = \frac{1}{2}$. It has only been improved since to $\log^{3/5}/\log\log^{1/5} = \log^{3/5-\epsilon}$ by Vinogradov and Korobov in 1958, and not at all since then. So, roughly, one can get $\alpha = 3/5 - \epsilon$ by analytic methods.
3. The best that has been done so far by elementary methods (*not* using Complex Analysis – see III.1) is $\alpha = 1/6 - \epsilon$ (Lavrik and Sobirov, 1973).
4. By Turán's method (above), this still yields a non-trivial zero-free region (though not, of course, as good as the classical one or the best-known one).

Primes in an Arithmetic Progression.

We consider primes in an arithmetic progression (AP, with first term h and common difference k – we may take h, k coprime). *Dirichlet's theorem (on primes in an AP)* states that there are *infinitely many* primes in any AP. This is Th. 15* in HW (where the proof is described as too difficult to include!) – though it is a very special case of the following, which says that there are "as many primes in an AP as there ought to be":

Theorem (Dirichlet). For $h, k \in \mathbb{N}$, $(h,k) = 1$, $x > 1$,

$$\pi(x;h,k) := \sum_{p \le x : p \equiv h \ (mod\ k)} \sim li(x)/\phi(k),$$

with ϕ the Euler totient function.

For proof, see e.g. J Ch. 4, R Ch. 13; this depends on *Dirichlet L-functions* (analogues of ζ). Error terms are known. But all this holds *uniformly* over many APs simultaneously – e.g., in $h, k \le (\log x)^u$. The uniformity plus the classical error term is the *Siegel-Walfisz theorem* (C. L. SIEGEL in 1935, A. WALFISZ in 1936). This can be strengthened further to the *Bombieri-Vinogradov theorem* (E. BOMBIERI 1965; A. I. VINOGRADOV 1965; this depends on the *large sieve* – K. ROTH, 1965). The theorem says that the strengthening obtainable here by assuming (RH) actually holds 'on average'.