m3pm16l4.tex
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§5. INFINITE PRODUCTS

For a formal infinite product $\prod_{n=1}^{\infty} u_n$, write

$$p_n := \prod_{k=1}^{n} u_k$$

for the nth *partial product*.
Defn. (i) If no factor u_n is 0, we say $\prod_1^\infty u_n$ *converges to* $p \neq 0$ if the sequence p_n converges to $p \neq 0$.
(ii) If finitely many u_n are 0, say $u_n \neq 0$ for $n > N$,

$$\prod_1^\infty u_n := u_1 \dots u_N \prod_{N+1}^\infty u_n$$

(convergent or divergent as in (i)).
(iii) If infinitely many $u_n = 0$, say $\prod_1^\infty$ *diverges to* 0.
(iv) $\prod_1^\infty u_n$ *diverges* if it does not converge as in (i) or (ii).

Cauchy criterion for products. As for sums: $\prod_1^\infty u_n$ converges iff

$$\forall \epsilon > 0 \ \exists N \text{ s.t. } \forall n \geq N \ \forall p \geq 0, |u_{n+1}. \dots .u_{n+p} - 1| < \epsilon.$$

Theorem. If each $a_n > 0$, $\prod(1 + a_n)$ converges iff $\sum a_n$ converges.

Proof. Write $s_n := a_1 + \ldots + a_n$, $p_n := (1 + a_1) \ldots (1 + a_n)$. Multiply out:

$$p_n = 1+a_1+\ldots+a_n+a_1a_2+\ldots > 1+a_1+\ldots+a_n = 1+s_n > s_n : \qquad p_n > s_n.$$

But $1 + x \leq e^x$ for $x \geq 0$, so taking $x = a_k$ and multiplying, $p_n \leq e^{s_n}$.
Combining, p_n bounded iff s_n bounded; each is increasing (as $a_n > 0$), so (as sequences) they converge or diverge together. As $p_n \geq 1$, if $p_n \to p$, then $p \geq 1$, so the sequence p_n cannot converge to 0. //

Defn. $\prod(1 + a_n)$ *converges absolutely* if $\prod(1 + |a_n|)$ converges.
As with sequences: absolute convergence implies convergence.

§6. THE RIEMANN-LEBESGUE LEMMA.

For $\phi : \mathbf{R} \to \mathbf{C}$ *integrable*, meaning

$$\int_{-\infty}^{\infty} |\phi(t)|dt < \infty,$$

define the *Fourier transform* $\hat{\phi}$ by

$$\hat{\phi}(\lambda) := \int_{-\infty}^{\infty} e^{i\lambda t}\phi(t)dt.$$

This exists, as $|e^{i\lambda t}\phi(t)| \leq |\phi(t)|$ and $\int |\phi| < \infty$.

Th. (Riemann-Lebesgue Lemma). If $\int |\phi| < \infty$ and ϕ has continuous derivative ($\phi \in C^1$), then

$$\hat{\phi}(\lambda) \to 0 \qquad (|\lambda| \to \infty).$$

Proof. Choose $\epsilon > 0$, and then take T so large that $\int_T^\infty |\phi| < \epsilon$, $\int_{-\infty}^{-t} |\phi| < \epsilon$. Then also $|\int_T^\infty e^{i\lambda t}\phi(t)dt| < \epsilon$, $|\int_{-\infty}^{T} e^{i\lambda t}\phi(t)dt| < \epsilon$ (as $|\int ...| \leq \int |...|$). As ϕ' is continuous on $[-T, T]$, it is bounded there, by M say. Write

$$\hat{\phi}_T(\lambda) := \int_{-T}^{T} e^{i\lambda t}\phi(t)dt.$$

Integrating by parts,

$$\hat{\phi}_T(\lambda) = \frac{1}{i\lambda}[e^{i\lambda t}\phi(t)]_{-T}^{T} - \frac{1}{i\lambda}\int_{-T}^{T} e^{i\lambda t}\phi'(t)dt.$$

So

$$|\phi_T(\lambda)| \leq \frac{1}{|\lambda|}(|\phi(T)| + |\phi(-T)|) + \frac{2TM}{|\lambda|} \to 0 \qquad (|\lambda| \to \infty).$$

So $|\phi_T(\lambda)| < \epsilon$ for $|\lambda|$ large enough. Adding in $\int_{-\infty}^{-T}$ and $\int_T^\infty$, $|\phi(\lambda)| \leq 3\epsilon$ for $|\lambda|$ large enough. //

Note. 1. We use here the *Riemann integral.* This suffices for this course, and you know it. The result is also true for the *Lebesgue integral* (more general, and easier to handle, so better, but harder to set up) – which not all of you know. With Lebesgue integrals, we do not need to assume ϕ' exists (or is continuous).

2. The Lebesgue integral is closely linked to *Lebesgue measure* (length, area, volume etc.). The general area is Measure Theory.