m3pm16l7.tex
Lecture 7. 29.1.2013

By far the most important Dirichlet series is that with $a_n \equiv 1$:
Defn. The *Riemann zeta function* is

$$\zeta(s) := \sum_{n=1}^{\infty} 1/n^s.$$

As we shall see (III), the properties of the zeta function, in particular the location of its zeros, are the key to the distribution of the primes, our main object of study.

For $\zeta(s)$: $\sigma_c = \sigma_a = 1$ ($\sum 1/n^c$ converges for $c > 1$, diverges for $c \leq 1$).

For the *Dirichlet eta function* (*alternating zeta function*)

$$\eta(s) := \sum_{n=1}^{\infty} (-)^{n-1}/n^s : \qquad \sigma_c = 0, \qquad \sigma_a = 1.$$

That $\sigma_c = 0$ follows as $\sum (-)^{n-1}/n^\sigma$ converges for $\sigma > 0$ by the Alternating Series Test, but diverges for $\sigma \leq 0$ (nth term does not tend to 0). That $\sigma_a = 1$ follows as taking the modulus inside the sum reduces $\eta(s)$ to $\zeta(s)$.
Note. So σ_a and σ_c may differ – indeed, by the maximum amount, 1.

Theorem (Analytic continuation of $\zeta(s)$). $\zeta(s)$ is (i.e. can be analytically continued to be) holomorphic in $Res > 0$, except for a simple pole at $s = 1$ of residue 1.

Proof (M2PM3 II.8 L23). With $\eta(s)$ as above,

$$\eta(\sigma) := \textstyle\sum_{n=1}^{\infty} (-)^{n-1}/n^\sigma = \displaystyle\sum_{odd} 1/n^\sigma - \sum_{even} 1/n^\sigma = \sum_o - \sum_e,$$

say. Now $\sum_e = \sum_1^\infty 1/(2n)^\sigma = 2^{-\sigma} \sum_1^\infty 1/n^\sigma = 2^{-\sigma}\zeta(\sigma)$. So

$$\textstyle\sum_{n=1}^{\infty} (-)^{n-1}/n^\sigma = \displaystyle\sum_o - \sum_e = \sum_o - 2^{-\sigma}\zeta(\sigma), \qquad \zeta(\sigma) = \sum_o + \sum_e = \sum_o + 2^{-\sigma}\zeta(\sigma).$$

Subtract: $\sum_1^\infty (-)^{n-1}/n^\sigma - \zeta(\sigma) = -2.2^{-\sigma}\zeta(\sigma) = -2^{1-\sigma}\zeta(\sigma)$:

$$\textstyle\sum_1^\infty (-)^{n-1}/n^\sigma = (1 - 2^{1-\sigma})\zeta(\sigma) : \quad \zeta(\sigma) = (1 - 2^{1-\sigma})^{-1}.\sum_1^\infty (-)^{n-1}/n^\sigma. \quad (*)$$

Note that the series on RHS converges in $\sigma > 0$, by the Alternating Series Test. Now let s be complex, and define

$$\zeta(s) := \eta(s)/(1 - 2^{1-s}) = (\textstyle\sum_{n=1}^{\infty} (-)^{n-1}/n^s)/(1 - e^{(1-s)\log 2}). \qquad (**)$$

As $\eta(s)$ has $\sigma_c = 0$, the first factor on RHS is holomorphic in $\sigma > 0$ (the second is holomorphic as $2^{-s} = e^{-s \log 2}$ is). So we may use $(*)$ to *continue* $\zeta(s)$ *analytically* from $\sigma > 1$ to $\sigma > 0$.

Near $s = 1$, $1 - 2^{1-s} = 1 - e^{(1-s)\log 2} = (s-1)\log 2 + O((s-1)^2)$. Now

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots + (-)^{n-1}/n + \ldots \qquad (|x| < 1).$$

As $x \downarrow -1$, LHS $\to -\log 2$; as $\sum (-)^{n-1}/n$ converges (above), Abel's Continuity Theorem (which we quote: see your favourite Analysis book) gives

$$\sum_{n=1}^{\infty} (-)^{n-1}/n = \log 2.$$

So as $s \to 1$, this common value cancels in $(**)$ to give $\zeta(s) \sim 1/(s-1)$: $\zeta(s)$ is holomorphic in $Re s > 0$ except for a simple pole at 1 of residue 1. //

Note. 1. Abel's Continuity Theorem goes from behaviour of the series (convergence) to behaviour of the corresponding power series (a smoothed version of it). There is no converse, but Tauber obtained a partial converse, under an additional condition. Such 'corrected converses' of 'Abelian theorems' are called *Tauberian theorems*. We shall use them in III.6, III.7 to prove PNT in III.8. We note here that *Littlewood's Tauberian theorem* (J. E. LITTLEWOOD (1885-1977) in 1911) concerns power series as here, under the Tauberian condition $a_n = O(1/n)$. The example above of the logarithmic series is at the limits of applicability of Littlewood's result.

2. In $(**)$, the denominator has zeros where

$$1 - 2^{1-s} = 0, \quad 2^{1-s} = e^{(1-s)\log 2} = 1 = e^{2\pi n i},$$

$$(1-s)\log 2 = 2\pi n i, \quad s = 1 + \frac{2\pi n i}{\log 2}, \qquad n \text{ integer}.$$

The case $n = 0$ is covered above. For $n \neq 0$, we shall see (III.3) that $\zeta(s)$ is holomorphic, so the zero of $1 - 2^{(1-s)}$ is cancelled by a zero of η.