m3pm16l8.tex
**Lecture 8. 29.1.2013**

## §2. Holomorphy

**Theorem**. A Dirichlet series is holomorphic within its half-plane of convergence, with derivative given by termwise differentiation. If $F(s) := \sum_1^\infty a_n/n^s$ for $\sigma > \sigma_c$, then $F'(s) = -\sum_1^\infty \log n \; a_n/n^s$.

*Proof.* Choose $\alpha > \sigma_c$ and write $b_n := a_n/n^\alpha$. As above, $b_n$ is bounded (by $M$, say). Write $F(s) = G(s-\alpha), G(s) := \sum_1^\infty b_n/n^s$. Write $G_N(s) := \sum_1^N b_n/n^s = \sum_1^N b_n e^{-s\log n}$. Then $G'_N(s) = -\sum_1^N \log n b_n/n^s$.

Take $\delta > 0, R > 0, K > 0, \Gamma$ the rectangle with sides $\sigma = \delta, \sigma = K, t = \pm R$, $E$ its interior. By II.1, $(**)$,

$$|G(s) - G_N(s)| \leq \frac{M}{N^\sigma}(\frac{|s|}{\sigma} + 1).$$

For $s \in E$,

$$\frac{|s|}{\sigma} \leq \frac{\sigma + |t|}{\sigma} = 1 + \frac{|t|}{\sigma} \leq 1 + \frac{R}{\sigma}.$$

So

$$|G(s) - G_N(s)| \leq \frac{M}{N^\delta}(2 + \frac{R}{\delta}) \to 0 \qquad (N \to \infty),$$

uniformly on $\Gamma \cup E$, which is compact. As each $G_N$ is holomorphic by I.2, $G$ is holomorphic. As each $s$ with $\sigma > 0$ is in some $E$, $G$ is holomorphic on $\sigma > 0$, so $F$ is holomorphic on $\sigma > \alpha$. Then $G'_N \to G'$ by I.2, so as $D(n^{-s}) = D(e^{-s\log n}) = -\log n \; n^{-s}$, $F'(s) = -\sum_1^\infty \log n \; a_n/n^s$. Similarly for Dirichlet integrals: if $I_X(s) := \int_1^X f(x)dx/x^{1+s}$, then $I'_X(s) = -\int_1^X f(x)\log x dx/x^{1+s}$ by differentiating under the integral sign. //

*Example.*

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}, \qquad \zeta'(s) = -\sum_1^\infty \log n/n^s \qquad (\sigma > 1).$$

By integrating by parts,

$$\int_1^\infty \frac{\log x}{x^\sigma} dx = \frac{1}{(\sigma-1)^2} \qquad (\sigma > 1).$$

Hence as in I.4,

$$-\zeta'(\sigma) \le \frac{1}{(\sigma-1)^2}.$$

## §3. Convolutions

As with power series, absolutely convergent series may be rearranged. So if

$$F_a(s) := \sum_1^\infty a_n/n^s, \qquad F_b(s) := \sum_1^\infty b_n/n^s,$$

then in the half-plane where both converge absolutely

$$F_a(s)F_b(s) = (\sum_{i=1}^\infty \frac{a_i}{i^s})(\sum_{j=1}^\infty \frac{b_j}{j^s}) = \sum_{ij} \frac{a_i b_j}{i^r j^s} = \sum_{n=1}^\infty \frac{c_n}{n^s},$$

where

$$c_n := \sum_{ij=n} a_i b_j = \sum_{i|n} a(i)b(n/i).$$

The series $c = (c_n)$ so obtained is called the *Dirichlet convolution* of $a$ and $b$:

$$c = a * b$$

(cf. I.6). Write $e_i := (\delta_{1n})$ (the Kronecker delta: 1 if $n = 1$, 0 otherwise). Then $a * e_1 = a$: $e_1$ acts as an identity.

Dirichlet convolutions have the properties:
$a * b = b * a$ – commutativity;
$a * (b + c) = a * b + a * c$ – distributivity;
$a * (b * c) = (a * b) * c$ – associativity.
Note also: $u := (u_n)$, where $u_n := 1$ for all $n$, so $u$ has Dirichlet series

$$\zeta(s) := \sum_1^\infty 1/n^s; \qquad (u, \zeta)$$

$d := (d_n)$, the *divisor function*, where $d_n := \sum_{d|n} 1$ is the number of divisors of $n$. Then

$$(u * u)_n = \sum_{d|n} u(d)u(n/d) = \sum_{d|n} 1 = d(n): \qquad u * u = d.$$

So one has the important Dirichlet series

$$\zeta(s)^2 = \sum_{n=1}^\infty d_n/n^s. \qquad (d, \zeta^2)$$