m3pm16mult.tex

MULTIPLICATION OF SERIES

Given two series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, write

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}.$$

Then $\sum_0^\infty c_n$ is called the *Cauchy product* of $\sum a_n$, $\sum b_n$.

Theorem. If $A := \sum_0^\infty a_n$ and $B := \sum_0^\infty b_n$ both converge absolutely, then their Cauchy product $C := \sum_0^\infty c_n$ converges absolutely, and

$$C = AB.$$

Proof. Write $A_n := \sum_0^n a_k$, and similarly for B_n, C_n. The terms of

$$A_n B_n = (a_0 + \ldots + a_n)(b_0 + \ldots + b_n)$$

can be written out in 'matrix form', as

$$\begin{array}{cccc} a_0b_0 & a_0b_1 & \ldots & a_0b_n \\ \vdots & \vdots & & \\ a_nb_0 & a_nb_1 & \ldots & a_nb_n. \end{array}$$

Now c_n is the sum of the terms on the 'backward diagonal' of this matrix, so $C := \sum_0^\infty c_n$ is the sum of *all* the terms, in the infinite matrix obtained as $n \to \infty$. Write $A'_n := \sum_0^n |a_k|$, and similarly for B'_n, C'_n. Now

$$C'_n = \sum_0^n |c_k| = \sum_0^n |\sum_0^k a_r b_{k-r}| \le \sum_{k=0}^{n} \sum_{r=0}^{k} |a_r||b_{n-r}|$$

$$\le (\sum_{i=0}^{n} |a_i|)(\sum_{j=0}^{n} |b_j|) = A'_n B'_n \le A'B' \qquad (*)$$

(in $(*)$, $\sum_{0\le r\le k\le n}$ contains *some* terms $|a_i||b_j|$ from the above matrix, $\sum_0^n \sum_0^n$ contains them *all*; $A'_n \le A'$, etc.) So C'_n is increasing, and bounded above by $A'B'$. So $C'_n \uparrow C' \le A'B'$: $\sum |c_n|$ is convergent. So $\sum c_n$ is absolutely convergent, to C say. So the terms in $C = \sum c_n$ may be rearranged:

$$C = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i,j\ge 0} a_i b_j = (\sum_{i=0}^{\infty} a_i)(\sum_{j=0}^{\infty} b_j) = AB. \qquad //$$

Note. It suffices for *one* of $\sum a_n$, $\sum b_n$ to be absolutely convergent (Mertens' theorem on multiplication of series). See J App. B, or Apostol, *Mathematical Analysis*, Th. 12.4.6.//

Generating Functions (GFs). Given a sequence $(a_n)_0^\infty$, write

$$A(x) := \sum_{n=0}^{\infty} a_n x^n$$

(this notation conflicts with that for sum-functions in Ch. II, but is useful for Ch. I and elsewhere in Mathematics). Then $A(x)$ is called the *generating function (GF)* of (a_n).

Recall that a power series is absolutely convergent inside its circle of convergence (and uniformly convergent on compact subsets of it).

Given (a_n), (b_n), with GFs $A(x)$, $B(x)$: within the smaller of their circles of convergence, both are absolutely convergent. So we may rearrange:

$$A(x)B(x) = (\sum_{i=0}^{\infty} a_i x^i)(\sum_{j=0}^{\infty} b_j x^j) = \sum_{i,j\geq 0} a_i b_j x^{i+j} = \sum_{n=0}^{\infty} x^n \sum_{i+j=n} a_i b_j = \sum_{n=0}^{\infty} c_n x^n = C(x),$$

with (c_n) the Cauchy product (or *convolution* of (a_n), (b_n).

Similar results hold for *Dirichlet series* and *Dirichlet products* (or *convolutions*) in place of power series and Cauchy products; see Ch. II.

Probability GFs (PGFs). If X is random variable taking values $0, 1, 2, \ldots$ only, write $a_n := P(X = n)$. Then

$$A(x) = \sum_0^{\infty} a_n x^n = \sum_0^{\infty} x^n P(X = n) = E[x^X]$$

is called the *probability GF (PGF)* of X. As $A(1) = \sum_0^\infty P(X = n) = 1 < \infty$, the radius of convergence (RC) R of a PGF is always ≥ 1. For $x < R$, we can differentiate to get

$$A'(x) = \sum_0^{\infty} n x^{n-1} P(X = n); \qquad A'(1) = \sum_0^{\infty} n P(X = n) = E[X],$$

and similarly for the higher derivatives (if $R = 1$, this still holds, but to show this we need *Abel's Continuity Theorem*, Ch. II). If X, Y are independent, with PGFs A, B, x^X, x^Y are independent, with PGF

$$C(x) := E[x^{X+Y}] = E[x^X.x^Y] = E[x^X]E[x^Y] \text{ (independence) } = A(x)B(x):$$

adding independent random variables $\Leftrightarrow$ *Cauchy product*, or *convolution*, of distributions, $\Leftrightarrow$ *multiplication* of PGFs.