m3pm16prob7.tex
(Week 8, because of Assessed Coursework, Week 7, deadline Week 9)

M3PM16/M4PM16 PROBLEMS 7. 7.3.2013

Q1 *(Mertens' Second Theorem extended to prime powers).* Show that Mertens' Second Theorem extends from primes p to prime powers p^n with a change of constant:

$$\sum_{p^n \leq x} 1/p^n = \log\log x + C_2 + O(1/\log x),$$

where

$$C_2 := C_1 + S, \qquad S := \sum_p \frac{1}{p(p-1)}.$$

Q2 *(Euler's formula for $\zeta(2)$ using only calculus).* Show that $\zeta(2) := \sum_1^\infty 1/n^2 = \pi^2/6$, by considering

$$I := \int_0^1 \int_0^1 \frac{dxdy}{1-xy}.$$

(Expand $1/(1-xy)$ as a power series and integrate term by term to show $I = \zeta(2)$. Change variables by $x = (u-v)/\sqrt{2}$, $y = (u+v)/\sqrt{2}$ to change from the unit square to the square S with opposite vertices $(0,0)$ and $(\sqrt{2}, 0)$. Divide S into four right-angled triangles. Use symmetry in $\pm u$ to reduce to the upper two, the left T_1 and the right T_2. Evaluate the integrals I_1, I_2 over the upper left and right triangles T_1, T_2 by finding the inner v-integrals as $\tan^{-1}$s and then the outer u-integrals by $u = \sqrt{2}\sin\theta$ in T_1 and $u = \sqrt{2}\cos 2\theta$ in T_2.

NHB