m3pm16prob8.tex
To Week 9 (Assessed Coursework, Week 7)

M3PM16/M4PM16 PROBLEMS 8. 14.3.2013

Q1. Show that

$$n! = \prod_p p^{j(n,p)}, \qquad j(n,p) := \sum_{m \geq 1} [n/p^m].$$

Q2. Show that

$$N := \binom{2n}{n} = \prod_{p \leq 2n} p^{k(p)}, \qquad k(p) = \sum_m ([2n/p^m] - 2[n/p^m]).$$

Q3. Show that $[2x] - 2[x]$ is 1 or 0 according as $[2x]$ is odd or even.
Deduce from Q1 that

$$k(p) \leq \left[\frac{\log\ 2n}{\log p}\right] \leq \frac{\log\ 2n}{\log p}.$$

Q4 (*Bertrand's postulate*, 1845). Show that
(i) for each $n \geq 1$ there is a prime p with $n < p \leq 2n$; equivalently,
(ii) if p_r is the rth prime, $p_{r+1} < 2p_r$.
(This was proved by Chebyshev in 1851, and by Erdös in 1932 by more elementary methods. There is a proof in HW §22.3 (which uses Q1-3 above) – see HW, or Solutions 8.)

NHB