m3pm16soln2.tex

M3PM16/M4PM16 SOLUTIONS 2. 2.2.2012

Q1.

$$\exp\{(\log x)^a\} = \sum_{n=0}^{\infty} (\log^a x)^n / n! = \sum_{0}^{\infty} (\log^{na} x)/n!$$

For any power $\log^k x$ of $\log x$, taking n so large that $na > k$ and letting $x \to \infty$ gives

$$\exp\{(\log x)^a\}/\log^k x \to \infty.$$

But $\log^a x < \log x$ as $a < 1$, so

$$\exp\{(\log x)^a\}/\exp\{\log x\} = \exp\{(\log x)^a\}/x \to 0.$$

Q2. (i) Then either form of remainder given is smaller than $O(x/\log^k x)$, so gives a better result in PNT.
(ii) Knowing that PNT with error term as in III.10.2 is available, this shows that it is simpler to use $li(x)$ throughout than $x/\log x$ or any of its logarithmic refinements.

Q3. If $n = 0$, take $x = y = 0$. So assume $n > 0$.
If (a, b) does not divide n, there is no solution ((a, b) divides LHS but not RHS).
If (a, b) divides n: by the Euclidean Algorithm, there are integers c, d with

$$ac + bd = (a, b).$$

So

$$a(\frac{nc}{(a, b)}) + b(\frac{nd}{(a, b)}) = n,$$

and as $(a, b)|n$, $(a, b) = mn$ say, this says

$$a(mc) + b(md) = n,$$

giving the required solution. //

Q4. As $(a, b) = 1$, there are integers m, n with

$$am + bn = 1,$$

by the Euclidean Algorithm. So

$$acm + bcn = c.$$

But $a|bc$, so $a|$LHS. So $a|$RHS, i.e. $a|c$. //

NHB