m3pm16soln4.tex

M3PM16/M4PM16 SOLUTIONS 4. 14.2.2013

Q1. (i) Using $|.|$ for cardinality,

$$\sum_{d|n} |a : 1 \leq a \leq n, (a, n) = d| = n,$$

as each integer a from 1 to n has a unique gcd with n, $d := (a, n)$, which divides n. Also, if $d|n$ then

$$\begin{aligned} \phi(n/d) &= |a : 1 \leq a \leq n/d, (a, n/d) = 1| \qquad \text{(definition of } \phi(n/d)\text{)} \\ &= |b : 1 \leq b \leq n, (b, n) = d| \qquad (b := da). \end{aligned}$$

Combining,

$$n = \sum_{d|n} \phi(n|d), = \sum_{d|n} \phi(d),$$

since as d runs through the divisors of n, so does n/d.
(ii) This follows by II.3 Propn. and (i).
(iii) For p^c, there are $p^c - 1$ positive integers $< p^c$, of which the multiples of p are $p, 2p, \ldots, p^c - p$ (so $p^{c-1} - 1$ of these), and the rest are coprime to p^c. So

$$\phi(p^c) = (p^c - 1) - (p^{c-1} - 1) = p^c(1 - \frac{1}{p}).$$

So if $n = \prod p^c$ is the prime-power factorisation of n (FTA), (ii) gives

$$\phi(n) = \prod \phi(p^c) = \prod p^c \prod (1 - \frac{1}{p}) = n \prod_{p|n} (1 - \frac{1}{p}).$$

Q2. (i) If $a \in A$ belongs to exactly m of the sets: if $m = 0$, a is counted in the RHS $S - S_1 + S_2 \ldots$ just once, in S itself. If $m > 0$, then a is counted: $1 = \binom{m}{0}$ times in S; $\binom{m}{1}$ times in S_1, …, $\binom{m}{r}$ times in S_r.
Altogether, a is counted

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} \ldots = (1 - 1)^m = 0$$

times. So the RHS is the cardinality of the set of points in none of the A_i.
(ii) The number of integers $\leq n$ and divisible by a is $[n/a]$.

If a is coprime to b, the number of integers $\leq n$ and divisible by both a and b is $[n/ab]$, etc.
So the number of integers $\leq n$ and not divisible by any of a coprime set of integers $a, b, \ldots$ is $[n] - \sum[n/a] + \sum[n/ab] \ldots$.
Taking $a, b, \ldots$ as the prime divisors of n,

$$\phi(n) = n - \sum \frac{n}{p} + \sum \frac{n}{pq} \ldots = n \prod_{p|n} \Big(1 - \frac{1}{p}\Big).$$

Q3 (see e.g. R. V. Churchill, *Fourier series and boundary value problems*, McGraw-Hill 1963, Ch. 4). Write a_n for the Fourier cosine coefficients of $|x|$ on $[-\pi, \pi]$ ($|.|$ is even, so we do not need sine terms). Then

$$\frac{1}{2}a_0 = \frac{1}{2}.\frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^{\pi} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos\ nx dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{n\pi}\int_0^{\pi} x d \sin nx$$

$$= \frac{2[x \sin nx]_0^{\pi}}{n\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nx dx = \frac{2}{n^2\pi}[\cos nx]_0^{\pi} = \frac{2(\cos n\pi - 1)}{n^2\pi}$$

$$= \frac{2((-1)^n - 1)}{n^2\pi} = -\frac{4}{\pi n^2}$$

if n is odd, 0 if n is even. So

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2m-1)x}{(2m-1)^2}.$$

Putting $x = 0$ gives

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{odd} 1/n^2 : \qquad \sum_{odd} = \pi^2/8.$$

But

$$\zeta(2) = \sum_1^{\infty} 1/n^2 = \sum_{odd} + \sum_{even} = \sum_{odd} + \frac{1}{4}\zeta(2) : \qquad \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}, \qquad \zeta(2) = \frac{\pi^2}{6}.$$

NHB