m3pm16soln7.tex

M3PM16/M4PM16 SOLUTIONS 7. 14.3.2013

Q2 (J 93-4, Prop. 2.6.5). Write $q := p^n$ for a generic prime power, and for primes p with $p^2 \leq x$, let r_p be the largest 'relevant power' (largest r with $p^r \leq x$). Then

$$\Delta := \sum_{q \leq x} 1/q - \sum_{p \leq x} 1/p = \sum_{p \leq \sqrt{x}} \sum_{r=2}^{r_p} 1/p^r.$$

But $\sum_2^\infty 1/p_r = 1/(p(p-1))$, summing the GP, so

$$\Delta \leq \sum_p \frac{1}{p(p-1)} = S$$

(above). Write

$$S_0 := \sum_{p \leq \sqrt{x}} \frac{1}{p(p-1)};$$

then

$$\begin{aligned} S - S_0 \leq \sum_{p > \sqrt{x}} &< \sum_{n > \sqrt{x}} \frac{1}{n(n-1)} \\ &= \frac{1}{\sqrt{[x]}} \qquad \left(\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \text{ sum telescopes}\right) \\ &\leq 2/\sqrt{x}. \end{aligned}$$

As $p^{r_p+1} \geq x$:

$$\sum_{r > r_p} \frac{1}{p^r} < \frac{1}{x}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \frac{1}{x(1-1/p)} \leq 2/\sqrt{x} \qquad (p \geq 2).$$

So

$$S_0 - \Delta = \sum_{p \leq \sqrt{x}} \sum_{r > r_p} 1/p^r < \pi(\sqrt{x}).2/x \leq 2/\sqrt{x}$$

($\pi(x) := \sum_{p \leq x} 1 \leq \sum_{n \leq x} 1 \leq x$). Combining, $S - \Delta \leq 4/\sqrt{x} = O(1/\log x)$. So the difference Δ in the sums here and in Mertens' Second Theorem is $S + O(1/\log x)$, and the result follows from Mertens' Second Theorem. //

Q2 (Tom M. Apostol: A proof that Euler missed: Evaluating $\zeta(2)$ the easy way. *Mathematical Intelligencer* **8** no. 1 (1983), 59-60;
W. J. LeVeque, *Topics in number theory*, Vol. 1, Addison-Wesley, 1956, p.122 Ex.6).

$$I := \int_0^1 \int_0^1 dxdy/(1-xy) = \int_0^1 \int_0^1 \sum_0^\infty x^n y^n dxdy = \sum_0^\infty \int_0^1 x^n dx \int_0^1 y^n dy$$

$$= \sum_0^\infty 1/(n+1)^2 = \sum_1^\infty 1/n^2 = \zeta(2).$$

The change of variable has Jacobian 1, and takes the bounding lines of the unit square to those of S, and $1 - xy = 1 - (u^2 - v^2)/2 = (2 - u^2 + v^2)/2$. So symmetry between $\pm u$ reduces I to

$$I = 4\int_0^{1/\sqrt{2}} \Big(\int_0^u \frac{dv}{2-u^2+v^2}\Big) du + 4\int_{1/\sqrt{2}}^{\sqrt{2}} \Big(\frac{dv}{2-u^2+v^2}\Big) du = I_1 + I_2,$$

say. Evaluating the inner integrals,

$$I_1 = 4\int_0^{1/\sqrt{2}} \tan^{-1}\Big(\frac{u}{\sqrt{2-u^2}}\Big)\frac{du}{\sqrt{2-u^2}}, \quad I_2 = 4\int_{1/\sqrt{2}}^{\sqrt{2}} \tan^{-1}\Big(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\Big)\frac{du}{\sqrt{2-u^2}}.$$

In I_1, $u = \sqrt{2}\sin\theta$, $du = \sqrt{2}\cos\theta d\theta = \sqrt{2-u^2}d\theta$, $\tan\theta = u/\sqrt{2-u^2}$, $\tan^{-1}(u/\sqrt{2-u^2}) = \theta$; the limits are $u = 0$, $\theta = 0$ and $u = 1/\sqrt{2}$, $\sin\theta = 1/2$, $\theta = \pi/6$. So $I_1 = 4\int_0^{\pi/6} \theta d\theta = 2(\pi/6)^2$.
In I_2, $u = \sqrt{2}\cos 2\theta$, $du = -2\sqrt{2}\sin 2\theta d\theta = -2\sqrt{2}\sqrt{1-\cos^2 2\theta}d\theta$
$= -2\sqrt{2}\sqrt{1-u^2/2}d\theta = -2\sqrt{2-u^2}d\theta$: $du/\sqrt{2-u^2} = -2d\theta$.

$$\frac{\sqrt{2}-u}{\sqrt{2-u^2}} = \frac{\sqrt{2}(1-\cos 2\theta)}{\sqrt{2-2\cos^2 2\theta}} = \frac{(1-\cos 2\theta)}{\sqrt{1-\cos^2 2\theta}} = \frac{1-\cos 2\theta}{\sqrt{(1-\cos 2\theta)(1+\cos 2\theta)}}$$

$$= \sqrt{\frac{(1-\cos 2\theta)}{1+\cos 2\theta)}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta : \qquad \tan^{-1}\Big(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\Big) = \theta.$$

The limits are $u = 1/\sqrt{2}$, $\cos 2\theta = 1/2$, $2\theta = \pi/3$, $\theta = \pi/6$ and $u = \sqrt{2}$, $\cos 2\theta = 1$, $\theta = 0$. So $I_2 = 4\int_{\pi/6}^0 \theta(-2d\theta) = 8\int_0^{\pi/6} \theta d\theta = 4(\pi/6)^2$. So $I = I_1 + I_2 = (2+4)(\pi/6)^2 = \pi^2/6$. //

NHB