m3pm16soln8.tex

M3PM16/M4PM16 SOLUTIONS 8. 21.3.2013

Q1. The numbers $1, 2, \ldots, n$ include: $[n/p]$ multiples of p; $[n/p^2]$ multiples of p^2; etc. Multiplying $1, 2, \ldots, n$ to get $n!$, the prime powers add, as required.

Q2. This follows from $N = (2n)!/(n!)^2$ and Q1.

Q3. If $[2x] = 2n+1$ is odd, $2n+1 \leq 2x < 2n+2$, $n+\frac{1}{2} \leq x < n+1$, $[x] = 2n$, so $[2x] - 2[x] = 2n + 1 - 2n = 1$. If $[2x] = 2n$ is even, $2n \leq 2x < 2n + 1$, $n \leq x < n + \frac{1}{2}$, $[2x] - 2[x] = 2n - 2.n = 0$.

In Q2, each term with $p^m > 2n$ is 0. There are $[\log\ 2n/\log p]$ other terms, each 0 or 1 by above, so $k(p) \leq [\log\ 2n/\log p] \leq \log 2n/\log p$.

Q4 (*Bertrand's postulate: Erdös' proof of 1932*; see HW §22.3).

We can check this by hand for $n \leq 2^9 = 512$ (below). For $n > 2^9$, we assume not and derive a contradiction. So, we assume there is no prime p with $n < p \leq 2n$.

Write $N := \binom{2n}{n}$. Then if p is a prime divisor of N, $p \leq 2n$, so $p \leq n$ by hypothesis. Also $k(p) \geq 1$ in Q2.

Assume $\frac{2}{3}n < p \leq n$ (we shall see that this case cannot occur). Then $2p \leq 2n < 3p$, $p^2 > \frac{4}{9}n^2 > n.\frac{4}{9}.512 > 2n$: $2n/p^2 < 1$, so $[2n/p^2] = 0$, so also $[2n/p^m] = 0$ for $m \geq 2$. So

$$k(p) := \sum_m ([2n/p^m] - 2[n/p^m]) = [2n/p] - 2[n/p].$$

As $2 \leq 2n/p < 3$ (above), $[2n/p] = 2$. As $1 \leq n/p < 3/2$ (above), $2[n/p] = 2$. Combining, $k(p) = 0$. But $k(p) \geq 1$ (above), so no such p exists.

We now have $p \leq \frac{2}{3}n$ for every prime factor p of N. So

$$\sum_{p|N} \log p \leq \sum_{p \leq \frac{2}{3}n} \log p = \theta(\frac{2}{3}n) \leq \frac{4}{3} \log 2.n, \qquad (1)$$

by Chebyshev's Upper Estimate (III.2).

If $k(p) \geq 2$, $2 \log p \leq k(p) \log p$, $\leq \log(2n)$ by Q3. So

$$\log p \leq \log \sqrt{2n} : \qquad p \leq \sqrt{2n}.$$

So there are at most $\sqrt{2n}$ such p. So

$$\sum_{k(p)\geq 2} k(p)\log p \leq \sqrt{2n}\log(2n)$$

(at most $\sqrt{2n}$ terms; in each, $p \leq 2n$, as $p|N = \binom{2n}{n}$). So by (1),

$$\log N = \sum_{p\leq 2n} k(p)\log p = \sum_{k(p)=1}\log p + \sum_{k(p)\geq 2} k(p)\log p$$

$$\leq \sum_{p|N}\log p + \sqrt{2n}\log(2n) \leq \frac{4}{3}\log 2.n + \sqrt{2n}\log(2n). \qquad (2)$$

Also, N is the largest term in the binomial expansion of $(1+1)^{2n} = 2^{2n}$, so $2^{2n} = 2 + \binom{2n}{1} + \ldots + \binom{2n}{2n-1} \leq 2nN$. So by (2) $2n\log 2 \leq \log(2n) + \log N \leq \frac{4}{3}\log 2.n + (1+\sqrt{2n})\log(2n)$, giving

$$2n\log 2 \leq 3(1+\sqrt{2n})\log(2n). \qquad (3)$$

Write

$$x := \frac{\log(n/512)}{10\log 2} > 0$$

(as $n > 512$): $10x\log 2 = \log(n/2^9)$,

$$10(1+x)\log 2 = \log(n/2^9) + \log(2^{10}) = \log(2n): \qquad 2n = 2^{10(1+x)}.$$

So (3) is $2^{10(1+x)}\log 2 \leq 3(1+2^{5(1+x)}.10(1+x)\log 2$:
$2^{10(1+x)} \leq 30(1+2^{5+5x})(1+x)$. Divide by $2^{10}.2^{5x}$:

$$\begin{aligned} 2^{5x} &\leq 30.2^{-5}(1+2^{-5-5x})(1+x) \leq 30.2^{-5}(1+2^{-5})(1+x) \qquad (x>0) \\ &< (1-2^{-5})(1+2^{-5})(1+x) \qquad (30.2^{-5} < 31.2^{-5} = 1-2^{-5}) \\ &< 1+x. \qquad (4) \end{aligned}$$

But $2^{5x} = \exp(5x\log 2) > 1 + 5x\log 2$ ($e^x > 1+x$ for $x>0$) $> 1+x$ ($5\log 2 = \log 2^5 > \log e = 1$). But this contradicts (4). So for each $n > 512$ there is indeed a prime p with $n < p \leq 2n$.

For the early primes: each of the primes

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631$$

is less than twice its predecessor. So for any $n \leq 630$, at least one such p satisfies $n < p \leq 2n$. Combining gives the result.

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