m3pmabeldir.tex

Handout (I.3): The Convergence Tests of Abel and Dirichlet

Dirichlet Test for Convergence: If a_n have bounded partial sums $A_n = \sum_1^n a_r$, and $v_n \to 0$, then $\sum a_n v_n$ converges.

Proof: If $|A_n| \leq K, |A_n v_n| \leq K v_n \to 0$.
In $\sum_0^{n-1} A_r(v_r - v_{r-1})$, $|A_r(v_r - v_{r-1})| \leq K(v_r - v_{r-1})$. As $v_n \to 0$, $\sum v_r - v_{r-1}$ is a convergent telescoping series, so $\sum A_r(v_r - v_{r-1})$ is convergent by the Comparison Test.
Combining, $\sum a_n v_n$ is convergent by Abel's Lemma. //

Abel's Test for Convergence. If $\sum a_n$ convergent and v_n is real, monotonic and convergent, then $\sum a_n v_n$ converges.

Proof: A_n is convergent, v_n is convergent, so $A_n v_n$ is convergent. A_n is also bounded, $A_n \leq K$. $\sum(v_r - v_{r-1})$ is a convergent telescoping series. As above, $\sum A_r(v_r - v_{r-1})$ converges by the Comparison Test. Combining, $\sum a_n v_n$ converges by Abel's Lemma. //

We include for reference some corollaries to Abel's Summation Formula. See [J] for details of proofs.

Corollary 1. (i) $\sum_{r \leq x} a_r f_r = A(x)f(x) - \int_1^x A(t)f'(t)dt$.
(ii) $\sum_{r \leq x} a_r(f(x) - f(r)) = \int_1^\infty A(t)f'(t)dt$.

Corollary 2. If $f \in C^1[2, x]$ and $a(1) = 0$, then $\sum_{2 \leq r \leq x} a_r f_r = A(x)f(x) - \int_2^x A(t)f'(t)dt$.

Proof: Take $y = 2$ and use $A(2) = a_1 + a_2 = a_2$. //

Corollary 3. If $f \in C^1[1, \infty]$, and $A(x)f(x) \to 0$ as $x \to \infty$, then $\sum_1^\infty a_r f_r = -\int_1^\infty A(t)f'(t)dt$, and then
$\sum_{r > q} a_r f_r = -A(x)f(x) - \int_x^\infty A(t)f'(t)dt$.

Proof: Take $y = 1$ and let $x \to \infty$. //