m3pm16cwsoln(14).tex

M3PM16/M4PM16 SOLUTIONS To ASSESSED COURSEWORK. 21.3.2014

Q1. $\prod_{p \le n} p \le 4^n$.

Proof (Erdös 1939, Kalmar 1939; cf. III.2, Chebyshev's Upper and Lower Estimates, L16, 18). This holds for $n = 2$; we take $n \ge 3$, and use induction. If n is even, n is not prime, so

$$\prod_{p \le n} p = \prod_{p \le n-1} p \le 4^{n-1} < 4^n,$$

completing the induction. If $n = 2m+1$ is odd,

$$\prod_{m+1 < p \le 2m+1} \text{ divides } \binom{2m+1}{m} = \frac{m(m+1)\ldots.(2m+1)}{m!},$$

as every prime p on the left divides the numerator on the right, but not the denominator. As $\binom{2m+1}{m} = \binom{2m+1}{m+1}$ and each appears in the binomial expansion of $(1+1)^{2m+1} = 2^{2m+1}$,

$$2\binom{2m+1}{m} < 2^{2m+1}: \qquad \binom{2m+1}{m} < \frac{1}{2}.2^{2m+1} = 4^m.$$

Combining,

$$\prod_{m+1 < p \le 2m+1} p < 4^m.$$

So

$$\prod_{p \le n} p = \prod_{p \le m+1} p.\prod_{m+1 < p \le 2m+1} p < 4^{m+1}.4^m = 4^{2m+1} = 4^n,$$

again completing the induction. //

Q2. With p_n the nth prime,

$$\prod_{p \le n} p < p_n^n.$$

By PNT (Problems 1 Q3), $p_n \sim n \log n$. So for all $\epsilon > 0$ there exists N_0 with $p_n < n^{1+\epsilon}$ for $n \ge N_0$. So

$$p_n^n < n^{n(1+\epsilon)} = e^{n(1+\epsilon)\log n} < e^{n(1+2\epsilon)} \qquad (n \ge N_1),$$

say. So (replacing ϵ by 2ϵ, as we may) for each $\epsilon > 0$ there exists $N(\epsilon)$ with

$$\prod_{p \le n} p < p_n^n < e^{n(1+\epsilon)} \qquad (n \ge N(\epsilon)).$$

NHB