M3PM16/M4PM16 EXAMINATION 2014

Q1. Recall that Euler's totient function $\phi(n)$ is defined to be the number of positive integers $\leq n$ coprime to n. Show that (with $u(n) \equiv 1$, $I(n) \equiv n$ and μ the Möbius function):

(i)

$$\sum_{d|n} \phi(d) = n : \qquad \phi * u = I;$$

(ii)

$$\phi = I * \mu : \qquad \phi(n) = \sum_{d|n} \mu(d).n/d = n \sum_{d|n} \mu(d)/d;$$

(iii) ϕ is multiplicative;

(iv) ϕ has Dirichlet series

$$\sum_{1}^{\infty} \phi(n)/n^s = \zeta(s-1)/\zeta(s);$$

(v)

$$\phi(n) = n \ \Pi_{p|n} \left(1 - \frac{1}{p}\right).$$

(You may use without proof any standard facts about Dirichlet convolutions and series, but these should be clearly stated.)

Q2. (i) State without proof Mertens' Second Theorem on sums of reciprocals of primes, $\sum_{p \leq x} 1/p$.

(ii) Deduce the corresponding estimate for reciprocals of prime powers, $\sum_{p^n \leq x} 1/p^n$.

Q3. (i) For the Riemann zeta function $\zeta(s) := \sum_1^\infty 1/n^s$ and the alternating zeta function $\eta(s) := \sum_1^\infty (-)^{n-1}/n^s$, state without proof the abscissae σ_c, σ_a of convergence and of absolute convergence.

(ii) By using η, or otherwise, show how to continue ζ analytically to $\sigma := Re\ s > 0$.

(iii) Show that η, ζ have no real zeros in $(0, 1)$.

(iv) Show that ζ has a simple pole at 1 of residue 1. (You may quote that $\sum_1^\infty (-)^{n-1}/n = \log 2$.)

(v) Given the functional equation for the Riemann zeta function in the form

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\frac{1}{2}(1-s))\zeta(1-s),$$

show that $\zeta(0) = -\frac{1}{2}$ and $\zeta(-2n) = 0$ $(n = 1, 2, \ldots)$.

(vi) Given further that ζ does not vanish on the 1-line $\sigma = 1$, show that all zeros of ζ other than these 'trivial zeros' $-2n$ lie in the critical strip $0 < \sigma < 1$.

Q4. Define ν by $\nu(n) := \mu(d)$ if $n = d^2$ is a square, 0 otherwise. Show that:

(i) $|\mu| = \nu * u$;

(ii) if $Q(x)$ is the number of square-free natural numbers $n \leq x$,
$[x] = \sum_{m \leq \sqrt{x}} Q(x/m^2)$;

(iii) $Q(x) = \sum_{m \leq \sqrt{x}} \mu(m)[x/m^2]$;

(iv)

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x})$$

(you may quote $\zeta(2) = \pi^2/6$).

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