M3P16 ANALYTIC NUMBER THEORY: SOLUTIONS TO EXAMINATION 2013

Q1 (i). Taking $x = p_n$ in $\pi(x) := \sum_{p \le x} 1$ gives $\pi(p_n) = \sum_{p \le p_n} 1 = n$. By PNT, $\pi(x) \sim x/\log x$, so $n \sim p_n/\log p_n$:

$$n \log p_n/p_n \to 1. \tag{1}$$

Taking logs of (1), $\log n + \log\log p_n - \log p_n \to 0$. Dividing this by $\log p_n$,

$$\frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \to 0.$$

But $\log x = o(x)$, so $\log\log p_n = o(\log p_n)$, so this says

$$\log n/\log p_n \to 1. \tag{2}$$

Multiply (1) and (2): $n \log n/p_n \to 1$, i.e. $p_n \sim n \log n$. // [**8**, seen]

(ii) With $d_n := p_{n+1} - p_n$, $D(x) := \sum_{1<n\le x} d_n = p_{[x]+1} - p_2 \sim p_{[x]+1} \sim x \log x$, by (i). Abel summation with $a_n := d_n$, $b(t) := 1/\log t$ $(t \ge 2)$ gives

$$\sum_{1<n\le x} \frac{d_n}{\log n} = \frac{D(x)}{\log x} + \int_2^x \frac{D(t)}{t \log^2 t} dt.$$

By above, $D(t) \sim t \log t$, so the first term on RHS $\sim x$. Also the integrand $\sim 1/\log t$, so the integral $\sim x/\log x$, so the second term is negligible w.r.t. the first term. So

$$\sum_{1<n\le x} \frac{d_n}{\log n} \sim x. \qquad [\mathbf{6}, \text{unseen}]$$

(iii) $\liminf d_n/\log n \le 1$: for if not, there exist $\delta > 0$ and N with

$$d_n/\log n \ge 1 + \delta > 1 \qquad \forall\ n \ge N.$$

Then

$$\sum_{1<n\le x} \frac{d_n}{\log n} = \sum_1^{N-1} + \sum_{N\le n\le x} ... \ge \sum_1^{N-1} ... + (1+\delta) \sum_{N \le n \le x} 1 :$$

$$\sum_{1<n\le x} \frac{d_n}{\log n} \ge const + (1+\delta)([x] - N + 1),$$

so

$$\liminf \frac{1}{x} \sum_{1<n\le x} \frac{d_n}{\log n} \ge 1 + \delta > 1,$$

contradicting $\lim ... = 1$. The same argument shows (by contradiction) that $1 \le \limsup ....$ [**6**, unseen]

[(i) seen, Prob. 1 Q4; (ii), (iii) unseen]

Q2, **Theorem (Mertens' Second theorem**; HW Th. 427).

$$\sum_{p\le x} 1/p = \log\log x + C_1 + O(1/\log x) \qquad (x \ge 2),$$

for some constant C_1.

Proof. We use Abel summation, with

$$a(n) := \log n/n \qquad (n \text{ prime}), \qquad 0 \text{ otherwise}, \qquad A(x) := \sum_{n\le x} a_n.$$

[**2**, seen]

By Mertens' First Theorem, $\sum_{p\le x} \log p/p = \log x + O(1)$ (given),

$$A(x) = \log x + r(x), \qquad |r(.)| \le c_0 \qquad (x > 1),$$

$a(1) = 0$, and

$$\sum_{p\le x} 1/p = \sum_{2\le n\le x} \frac{a(n)}{\log n}. \qquad [\mathbf{4}, \text{seen}]$$

By Abel summation, this gives

$$\sum_{p\le x} 1/p = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t\ \log^2 t} dt = 1 + \frac{r(x)}{\log x} + \int_2^x \frac{dt}{t\log t} + I(x), \quad [\mathbf{4}, \text{seen}]$$

where

$$I(x) := \int_2^x \frac{r(t)}{t\ \log^2 t} dt. \qquad [\mathbf{2}, \text{seen}]$$

But

$$\int_2^x \frac{1}{t\log t} dt = \log\log x - \log\log 2, \qquad [\mathbf{2}, \text{seen}]$$

$$\int_2^\infty \frac{dt}{t\ \log^2 t} < \infty, \qquad \text{as} \qquad \frac{1}{t\ \log^2 t} = -\frac{d}{dt}\Big(\frac{1}{\log t}\Big). \qquad [\mathbf{2}, \text{seen}]$$

So $I(x) \to I$, finite, as $x \to \infty$, and

$$I(x) = I - s(x), \qquad |s(x)| \le c_0 \int_x^\infty \frac{dt}{t\ \log^2 t} = \frac{c_0}{\log x}. \qquad [\mathbf{2}, \text{seen}]$$

This gives the result with $C_1 := 1 - \log\log 2 + I$. // [**2**, seen]
[Seen, Lecture 13]

Q3. The *Möbius function* μ is defined by $\mu(1) := 1$; $\mu(n) := (-)^k$ is n is a product of k distinct primes; $\mu(n) = 0$ otherwise (i.e., if n is not square-free). [**1**, seen]

The *von Mangoldt function* Λ is defined by $\Lambda(n) = \log\ p$ if $n = p^m$ is a prime power, 0 otherwise. [**1**, seen]

(i) For $n = 1$, $u(1)\mu(1) = 1.1 = 1$; for $n > 1$, $(u * \mu)(n) := \sum_{i|n} \mu(i)$. If $n = p_1^{r_1} \dots p_k^{r_k}$ (from FTA), the $i > 1$ with $\mu(i) \neq 0$ are of the form $i = q_1 \dots q_j$ with the qs distinct primes from $\{p_1, \dots, p_k\}$. There are $\binom{k}{j}$ such choices, each giving an i with $\mu(i) = (-)^j$. As $\binom{n}{0} = 1$, this holds also for $j = 0$. So by the Binomial Theorem,

$$(u * \mu)(n) = \sum_{i|n} \mu(i) = \sum_{j=0}^{k} (-)^j \binom{k}{j} = (1-1)^k = 0. \qquad [\mathbf{4}, \text{seen}]$$

(ii) The same proof gives

$$\sum_{i|n} |\mu(i)| = \sum_{j=0}^{k} \binom{k}{j} = (1+1)^k = 2^k. \qquad [\mathbf{2}, \text{seen}]$$

(iii) With $\ell(n) := \log n$, $\Lambda(1) = \ell(1) = 0$. For $n > 1, n = p_1^{r_1} ... p_k^{r_k}$, say. Then $(\Lambda * u)(n) = \sum_{i|n} \Lambda(i)$. The divisors i of n are $i = p_1^{s_1} ... p_1^{s_k}, 0 \leq s_j \leq r_j$. Those with $\Lambda(i) \neq 0$ are those with $i = p_j^{s_j}$, $1 \leq j \leq k$, each with $\Lambda(i) = \log p_j$. There are r_j possibilities for each j, so $\sum_{i|n} \Lambda(i) = \sum_{j=1}^{k} r_j \log p_j = \log \prod_j p_j^{r_j} = \log n = \ell(n)$. So $\Lambda * u = \ell$, and then $\ell * \mu = \Lambda$ follows by Möbius inversion (here $u = (u_n)$, $u_n = 1$ for all n), i.e. $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d$. // [**4**, seen]

(iv) On the left: the Dirichlet series of $1/\zeta$ is $\sum_1^\infty \mu(n)/n^s$. Differentiating, that of $D(1/\zeta)$ is $-\sum_1^\infty \mu(n) \log n/n^s$. On the right: the Dirichlet series of $-\zeta'/\zeta$ is $\sum_1^\infty \Lambda(n)/n^s$. Since the product of Dirichlet series is the Dirichlet series of the Dirichlet convolution, (iv) follows. [**4**, unseen]

(v) Now (iv) says $-(\mu\ell) = \mu * \Lambda$. So by Möbius inversion, $\Lambda = -(\mu\ell) * u$:

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d. \qquad [\mathbf{4}, \text{unseen}]$$

[(i)-(iii) seen, Lectures 10-12, II.6,7; (iv), (v) unseen – HW Th. 297, 298.]

Q4. (i) $3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = 2(1+\cos\theta)^2$. // [**2**, seen]

(ii) If all $a_n \geq 0$ and the Dirichlet series $f(s) := \sum_1^\infty a_n/n^s$ converges for $Re\ s = \sigma > \sigma_0$, then

$$3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0 \qquad (\sigma > \sigma_0).$$

Proof. $3f(\sigma)+4Ref(\sigma+it)+Ref(\sigma+2it) = \sum_1^\infty \frac{a_n}{n^\sigma} Re(3+4n^{-it}+n^{-2it})$. If $\theta_n := t\log n$, $Re(3+4n^{-it}+n^{-2it}) = 3+4\cos\theta_n + \cos 2\theta_n \geq 0$, and $a_n/n^\sigma \geq 0$. So the sum of their products is ≥ 0. // [**5**, seen]

Hence, for $\sigma > 1$ and all t,

$$H(\sigma) := \zeta(\sigma)^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \geq 1.$$

Proof. $\log\zeta(s)$ has a Dirichlet series with non-negative coefficients, $\log\zeta(s) = f(s) = \sum_1^\infty a_n/n^s$ for $a_n \geq 0$ ($a_n = 1/m$ if $n = p^m$ is a prime power, 0 otherwise). By the Proposition, $3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0$. So ($\log z = \log(re^{i\theta}) = \log r + i\theta$, so $Re\log z = \log r = \log|z|$)

$$3\log\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \geq 0.$$

Exponentiating gives the result. // [**5**, seen]

Hence: $\zeta(1+it) \neq 0$ for $t \neq 0$.

Proof (by contradiction). If not, $\zeta(1+it) = 0$ for some $t \neq 0$. Then

$$\frac{\zeta(\sigma+it) - \zeta(1+it)}{(\sigma+it) - (1+it)} = \frac{\zeta(\sigma+it)}{\sigma - 1} \to \zeta'(1+it) \qquad (\sigma \downarrow 1),$$

as ζ is holomorphic at $1+it$ (indeed, everywhere except at 1 – proved in lectures). In the Corollary,

$$H(\sigma) = [(\sigma-1)\zeta(\sigma)]^3 . \left(\frac{|\zeta(\sigma+it)|}{\sigma-1}\right)^4 .(\sigma-1).[|\zeta(\sigma+2it)|].$$

Now $(\sigma-1)\zeta(\sigma) \to 1$ $(\sigma \downarrow 1)$ (ζ has a simple pole of residue 1 at 1). So $[...]^3 \to 1$; $(...)^4 \to (\zeta'(1+it))^4$ by above; $|\zeta(\sigma+2it)| \to \zeta(1+2it)$. Combining, the third factor $\sigma - 1$ gives $H(\sigma) \to 0$ as $\sigma \to 1$, contradicting the Corollary above. // [**6**, seen]

This result is needed to ensure the holomorphy on the 1-line of $-\zeta'/\zeta$, whose Dirichlet series $\sum_1^\infty \Lambda(n)/n^s$ encodes PNT. [**2**, seen]
[Seen: Lecture 20, III.4]