m3pm16l11.tex
Lecture 11. 7.2.2014

Möbius Inversion
Corollary 1.
$b(n) = \sum_{i|n} a(i)$, i.e. $b = a * u, \Leftrightarrow a(n) = \sum_{i|n} \mu(i) b\left(\frac{n}{i}\right)$, i.e. $a = b * \mu$.

Proof. If $b = a * u$, then $b * \mu = a * u * \mu = a * (u * \mu) = a * e_1 = a$. Similarly, if $a = b * \mu$, then $a * u = b * \mu * u = b * e_1 = b$. //

Note. Möbius inversion is important in Combinatorics. See e.g. Ch. 12 of P.J Cameron: *Combinatorics: Topics, Techniques, Algorithms*, CUP 1999.

Corollary 2. If F vanishes near 0, and $G(x) := \sum_1^\infty F(x/n)$ for $x > 0$, then $F(x) = \sum_1^\infty \mu(n) G(x/n)$.
Proof. As F is 0 near 0, the sum for G is finite. Then

$$\begin{aligned} F(x) &= \sum_1^\infty e_1(j) F(x/j) \qquad (e_1(j) = \delta_{1j}, = 1 \text{ as } j > 1) \\ &= \sum_1^\infty F(x/j) \sum_{n|j} \mu(n) \qquad (\mu * u = e_1) \\ &= \sum_{n=1}^\infty \mu(n) \sum_{k=1}^\infty F(x/kn) = \sum_1^\infty \mu(n) G(x/n). \qquad // \end{aligned}$$

Note. Since $1/\zeta(s) = \sum_1^\infty \mu(n)/n^s$ for $\sigma > 1$, and $\zeta(\sigma) \to \infty$ as $\sigma \to 1$, one would expect that $1/\zeta(1) = \sum_1^\infty \mu(n)/n = 0$. This is true, but equivalent to PNT (see III.10.4, 2012 – link on website; [A] Ch. 4, [R], §13.2). The sum function $M(x) := \sum_{n \le x} \mu(n)$ is also important. We shall see later that PNT implies that $M(x) = o(x)$. Indeed, PNT is also equivalent to it (III.10.4, 2012). Meanwhile, we estimate the partial sums.

Prop. $|\sum_{n=1}^N \mu_n/n| \le 1$ for all N.

Proof. As $\mu * u = e_1$ and $u_n \equiv 1$, writing $\{.\}$ for the fractional part,

$$1 = \sum_1^N (\mu * u)(n) = \sum_1^N \mu_n \sum_{n|N} 1 = \sum_1^N \mu_n [N/n] = \sum_1^N \mu_n ((N/n) - \{N/n\}) = N \sum_1^N \mu_n/n - r_N,$$

where $r_N := \sum_1^N \mu_n\{N/n\}$. As $\{N/1\} = \{N\} = 0$, $|r_N| = |\sum_2^N \mu_n\{N/n\} \leq \sum_2^N |\mu_n| \leq N - 1$. Combining, $N|\sum_1^N \mu_n/n| \leq 1 + (N-1) = N$. //

In fact, $\sum_1^\infty \mu_n/n$ converges to 0. This looks obvious, as this is $1/\zeta(s)$ for $s = 1$, $\zeta(s) = +\infty$ for $s = 1$, and $\zeta(s).1/\zeta(s) \equiv 1$. But this is in fact equivalent to PNT!

7. More Special Dirichlet Series

Squares and square-free numbers. Write S for the set of *squares* n^2: $I_S(n) := 1$ if $n \in S$, 0 otherwise, Q for the set of *square-free* numbers (no square factors: 'quadratfrei' in German).

$$\zeta(2s) = \sum_1^\infty 1/n^{2s} = \sum_1^\infty 1/(n^2)^s = \sum_1^\infty I_S(n)/n^s. \qquad (I_S)$$

If a is completely multiplicative with $|a_n| < 1$ and $\sum |a_n| < \infty$, write $S_1 := \sum_1^\infty a_n$, $S_2 := \sum_1^\infty a_n^2$. Then (Euler products, II.4 L9)

$$S_1/S_2 = \prod_p \frac{1}{1-a_p} / \prod_p \frac{1}{1-{a_p}^2} = \prod_p \frac{1-{a_p}^2}{1-a_p} = \prod_p (1 + a_p).$$

Expanding the RHS, we get a sum over a_n with n *square-free* (only *distinct* prime factors occur). So $S_1/S_2 = \sum_n |\mu(n)| a_n = \sum_n \mu(n)^2 a_n$ ($|\mu(n)| = \mu(n)^2 = 1$ if n is square-free, 0 otherwise). Taking in particular $a_n = 1/n^s$:

$$\zeta(s)/\zeta(2s) = \sum_1^\infty |\mu(n)|/n^s = \sum_1^\infty \mu(n)^2/n^s = \sum_{n=1}^\infty I_Q(n)/n^s \quad (Re\ s > 1). \qquad (\mu^2)$$

Cor. For $s = \sigma + it$, $\sigma > 1$:

$$\left|\frac{1}{\zeta(s)}\right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)} \leq \zeta(\sigma); \qquad \left|\frac{1}{\zeta(s)} - 1\right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)} - 1 \leq \zeta(\sigma) - 1.$$

Proof. $|1/\zeta(s)| = |\sum_1^\infty \mu_n/n^s| \leq \sum_1^\infty |\mu(n)/n^s| \leq \sum_1^\infty |\mu(n)|/n^\sigma = \zeta(\sigma)/\zeta(2\sigma)$ (above) $\leq \zeta(\sigma)$ ($\zeta(2\sigma) \geq 1$). Similarly for the second, subtracting the 1. //

Euler's totient function, $\phi(n) := \#\{r \leq n : (r, n) = 1\}$. See Problems 4.