m3pm16l13.tex

Lecture 13. 11.2.2014

Now use the Integral Test argument (I.4) to estimate $S(x)$. As

$$\int_1^x \log t dt = x\log x - x + 1$$

(integrate by parts), this gives

$$S(x) = x\log x - x + b(x), \qquad |b(x)| \leq \log x + 1.$$

Now $\log x + 1 \leq x$ for $x > 1$ (integrate $1/x \leq 1$ over $[1, x]$). So $|b(x)| \leq x$. So

$$x\sum_{n\leq x}\Lambda(n)/n = S(x) + a(x) = x\log x - x + a(x) + b(x).$$

But $0 \leq a(x) \leq 2x$, $|b(x)| \leq x$, so $|a(x) - x + b(x)| \leq 2x$. //

Cor.

$$\int_1^x \frac{\psi(t)}{t^2}dt = \log x + O(1) \qquad (x > 1).$$

Proof. Integrating by parts (or by Abel summation (I.3)),

$$\sum_{n\leq x}\Lambda(n)/n = \int_1^x d\psi(x)/x = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2}dt.$$

But $\psi(x)/x$ is bounded (from Chebyshev's θ-function: III.2), so this follows from the Theorem. //

The next result shows that we can neglect the powers of primes in the Theorem (at the cost of losing the bound 2): *powers of primes become sparse*, so this is not too surprising.

Theorem (Mertens' First Theorem: F. MERTENS (1840-1927) in 1874; HW Th. 425).

$$\sum_{p\leq x}\log p/p = \log x + O(1) \qquad (x > 1) \qquad (|O(.)| \leq 4).$$

Proof. As $\Lambda(n) = \log p$ when $n = p^m$,

$$0 \leq \sum_{n\leq x}\Lambda(n)/n - \sum_{p\leq x}\log p/p = \sum_{p^m\leq x}\log p/p - \sum_{p\leq x}\log p/p$$

$$= \sum_{m \geq 2} \sum_{p^m \leq x} \log p/p \leq \sum\nolimits_{p \leq x} \log p \Big(\frac{1}{p^2} + \frac{1}{p^3} + \dots \Big)$$

Summing the geometric series, the RHS is

$$\sum\nolimits_{p \leq x} \frac{p}{p(p-1)} \leq \sum\nolimits_{n=2}^{\infty} \frac{\log n}{n(n-1)} < \infty$$

(convergent, to sum ≤ 2 – check), giving the result by the Theorem above. //

Theorem (Mertens' Second theorem; HW Th. 427).

$$\sum\nolimits_{p \leq x} 1/p = \log\log x + C_1 + O(1/\log x) \qquad (x \geq 2),$$

for some constant C_1.

Proof (Compare $\sum_{n \leq x} 1/n = \log x + \gamma + o(1)$, I.4). Write

$$a(n) := \log n/n \qquad (n \text{ prime}), \qquad 0 \text{ otherwise}, \qquad A(x) := \sum\nolimits_{n \leq x} a_n$$

(so $a(1) = 0$). By Mertens' First Theorem,

$$A(x) = \log x + r(x), \qquad |r(.)| \leq c_0 \qquad (x > 1),$$

$$\text{and} \quad \sum\nolimits_{p \leq x} 1/p = \sum\nolimits_{2 \leq n \leq x} \frac{a(n)}{\log n} = \int_2^x dA(u)/\log u.$$

Integrating by parts (or by Abel summation), this gives

$$\sum\nolimits_{p \leq x} 1/p = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t \log^2 t} dt = 1 + \frac{r(x)}{\log x} + \int_2^x \frac{dt}{t \log t} + I(x),$$

$$I(x) := \int_2^x \frac{r(t)}{t \log^2 t} dt.$$

But

$$\int_2^x \frac{1}{t \log t} dt = \log\log x - \log\log 2, \qquad \int_2^{\infty} \frac{dt}{t \log^2 t} < \infty, \quad \text{as} \quad \frac{1}{t \log^2 t} = -\frac{d}{dt}\Big(\frac{1}{\log t}\Big).$$

So $I(x) \to I$, finite, as $x \to \infty$, and

$$I(x) = I - s(x), \qquad |s(x)| \leq c_0 \int_x^{\infty} \frac{dt}{t \log^2 t} = \frac{c_0}{\log x}.$$

This gives the result with $C_1 := 1 - \log\log 2 + I$. //