m3pm16l14.tex

Lecture 14. 14.2.2014

Theorem(Merten's Formula, HW Th 929).

$$\prod_{p\le x}(1-\frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x} \qquad (x\to\infty).$$

Proof. Write $\sum := \sum_p \left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$, which is convergent. By Merten's Second Theorem and the Constants Lemma (from the Website),

$$\sum_{p\le x}\frac{1}{p} = \log\log x + C_1 + o(1) = \log\log x + \gamma + \Sigma + o(1).$$

Now,

$$\sum_{p\le x}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) = \Sigma + o(1),$$

from the definition of Σ. Subtracting:

$$\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = -\log\log x - \gamma + o(1).$$

That is,

$$\log\left[\prod_{p\le x}\left(1-\frac{1}{p}\right)\right] = \log\left[\frac{e^{-\gamma}}{\log x}\right] + o(1).$$

So

$$\log\left[\prod_{p\le x}(1-\frac{1}{p})/\frac{e^{-\gamma}}{\log x}\right] \to 0 \qquad (x\to\infty).$$

So [...] $\to 1$, i.e.

$$\prod_{p\le x}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \qquad //$$

Note. Various attempts have been made to generalise multiplicative number theory beyond the primes p. Most of these get the orders of magnitude right. What they do not get right is the $e^{-\gamma}$ in Mertens' formula. See various attempts to introduce randomness (see the Handout 'Prime divisor functions; Landau's Poisson extension of PNT; probabilistic number theory').

9. Dirichlet's Hyperbola Identity (DHI)

Theorem (DHI). If $1 < y < x$,

$$\sum_{n \le x} (a * b)(n) = \sum_{j \le y} a(j) B(x/j) + \sum_{k \le x/y} b(k) A(x/k) - A(y) B(x/y).$$

Proof. LHS $= S := \sum_{jk \le x} a_j b_k$, as in II.3. Write S_1 for the sum of all such terms with $j \le y$, S_2 that of all terms with $k \le x/y$. As in II.3,

$$S_1 = \sum\nolimits_{jk \le x, j \le y} a_j b_k = \sum\nolimits_{j \le y} a_j \sum\nolimits_{k \le x/j} b_k = \sum\nolimits_{j \le y} a_j B(x/k),$$

the first sum on RHS, and similarly

$$S_2 = \sum\nolimits_{jk \le x, k \le x/y} a_j b_k = \sum\nolimits_{k \le x/y} b_k \sum\nolimits_{j \le x/k} a_j = \sum\nolimits_{k \le x/y} b_k A(x/k),$$

the second sum on RHS. Now $S_1 + S_2$ counts all terms, but counts twice those with both $j \le y$ and $k \le x/y$. The sum of these terms is $A(y)B(x/y)$. So subtracting this 'corrects the count', and gives the result. //

Theorem. If d_n is the number of divisors of n,

$$\sum_{n \le x} d_n = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Take $a_n = b_n = 1$ (so $(a * b)_n = d_n$, by (i)), $y = \sqrt{x}$: as $A(x) = B(x) = [x]$, Dirichlet's Hyperbola Identity gives

$$\sum_{n \le x} d_n = \sum_{j \le \sqrt{x}} [x/j] + \sum_{k \le \sqrt{x}} [x/k] - [\sqrt{x}][\sqrt{x}] = 2 \sum_{j \le \sqrt{x}} [x/j] - [\sqrt{x}][\sqrt{x}].$$

In each [.] on RHS, write $[.] = . - \{.\}$. Each fractional part $\{.\} \in [0, 1)$, so

$$\sum_{n \le x} d_n = 2 \sum_{j \le \sqrt{x}} x/j + O(\sqrt{x}) - x + O(\sqrt{x}) = 2x \sum_{j \le \sqrt{x}} 1/j - x + O(\sqrt{x}),$$

as $(\sqrt{x} + O(1))^2 = x + O(\sqrt{x})$. But as in L3, I.4,

$$\sum\nolimits_{j \le \sqrt{x}} 1/j = \log \sqrt{x} + \gamma + O(1/\sqrt{x}) = \frac{1}{2} \log\ x + \gamma + O(1/\sqrt{x}).$$

So

$$\sum_{n \le x} d_n = 2x(\log \sqrt{x} + \gamma + O(1/\sqrt{x})) - x + O(\sqrt{x}) = x \log x + (2\gamma - 1)x + O(\sqrt{x}). \ //$$