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III. THE PRIME NUMBER THEOREM AND ITS RELATIVES

§1. The Prime Number Theorem (PNT)

PNT states that

$$\pi(x) := \sum_{p \le x} 1 \sum li(x) := \int_2^x dt/\log t \sim x/\log x \qquad (x \to \infty) \quad (PNT)$$

This was conjectured on numerical grounds by GAUSS (c. 1799; letter of 1848) and A. M. LEGENDRE (1752-1833; in 1798, *Essai sur la Théorie des Nombres*).

In 1737 L. EULER (1707-1783) found his Euler product, linking the primes to $\sum_{n=1}^{\infty} 1/n^{\sigma}$ for real σ (later the Riemann zeta function).

In 1850-51, P. L. CHEBYSHEV (= TCHEBYSHEV, etc., 1821-1894) made two great strides (III.2):
(i) $\pi(x) \asymp x/\log x$, i.e. $C_1 x/\log x \le \pi(x) \le C_2 x/\log x$ for some $0 < C_1 \le C_2 < \infty$ and all $x \ge X$;
(ii)

$$\liminf \pi(x)/\frac{x}{\log x} \le 1 \le \limsup \pi(x)/\frac{x}{\log x}$$

– so if the limit exists (which we shall prove!), it must be 1.

In 1859 B. RIEMANN (1626-66) studied

$$\zeta(s) := \sum_{n=1}^{\infty} 1/n^s \qquad (s \in \mathbb{C})$$

using Complex Analysis (M2PM3), then still fairly new, developed by A. L. CAUCHY (1789-1857), 1825-29. He showed the critical relevance of the *zeros of* $\zeta(s)$ to the *distribution of primes*. We shall show that:
(i) ζ can be continued analytically from $Re\ s > 1$ to the whole complex plane $\mathbb{C}$, where it is holomorphic except for a simple pole at 1 of residue 1 (III.3);
(ii) The only zeros of ζ outside the *critical strip*

$$0 < \sigma = Re\ s < 1$$

are the so-called *trivial zeros* $-2, -4, \ldots, -2n, \ldots$ (trivial in that they follow from the *functional equation* for ζ – see III.7);
(iii) PNT is closely linked to non-vanishing of ζ on the 1-line (III.4):

$$\zeta(1 + it) \neq 0.$$

Indeed, PNT is equivalent to this.

The *Riemann Hypothesis* (RH) of 1859 is that the only zeros of ζ in the critical strip are on the *critical line*

$$\sigma = \frac{1}{2}. \tag{RH}$$

RH is still open, and is the most famous and important open question in Mathematics. Its resolution would have vast consequences for prime-number theory (especially error terms in PNT – see e.g. J Ch. 5). It is so hard that proving theorems *conditional on RH* (i.e., assuming it is true) is respectable in Analytic Number Theory.

PNT was proved independently in 1896 by J. HADAMARD (1865-1963, French) and Ch. de la Vallée Poussin (1866-1962, Belgian). Both used Complex Analysis and ζ.

Since counting primes relates to $\mathbb{N}$ ($\subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$), it seemed strange and unaesthetic to use complex methods. Great efforts were make to provide an *elementary proof*, over half a century.

Elementary proofs of PNT were found in 1948 by Paul ERDÖS (1913-1996, Hungarian) and Atle SELBERG (1917-2009, Norwegian). There is a full account by J. Spencer and R. Graham in *The Mathematical Intelligencer*, **31**.3 (2009), 18-23. Erdös gave an elementary proof of a result of Chebyshev (proof of Bertand's postulate: Problems 8). Selberg told him the day after seeing it that he could use it to complete an elementary proof of PNT. Erdös proposed collaboration but Selberg declined; their papers were published separately in 1949.

Proofs of ANT by complex methods are in all the books on ANT, including J Ch. 3, A, R, D. Elementary proofs of PNT are harder; see e.g. HW Ch. XXII (22.14-16), J Ch. 6, A Ch. 4, R Ch. 13.

Error estimates in PNT are very important. Naturally, complex methods give better error estimates than elementary ones. Error estimates depend on *zero-free regions* of ζ (to the left of the 1-line, in the critical strip) – the bigger, the better; see IV.3.