m3pm16l17.tex

Lecture 17. 21.2.2014

Proof of Chebyshev's Upper Estimate, continued.

Let $p_{k+1}, ...p_m$ be the primes with $n+2 \le p \le 2n+1$, so $\sum_{k+1}^m \log p_j = \theta(2n+1) - \theta(n+1)$. By (ENT1), no such p divides $n!$, but each divides $(2n+1)...(n+2) = n!N$. So by (ENT1), each divides N, and by (ENT2) their product divides N, so is $\le N$. So

$$\theta(2n+1) - \theta(n+1) = \log(p_{k+1}...p_m) \le \log N < n \log 4. \qquad (*)$$

We now show by induction that $\theta(n) \le n \log 4 \qquad (n \ge 2)$.
The induction starts, as $\theta(2) = \log 2 \le 2 \log 4$.
Assume that the condition holds for all $k \le 2n$, for $n \ge 1$.
Then in particular, $\theta(n+1) \le (n+1) \log 4$, but we have by $(*)$:

$$\theta(2n+1) \le (2n+1) \log 4.$$

Also, $\theta(2n+2) = \theta(2n+1)$, as $2n+2$ is not prime. So

$$\theta(2n+2) \le 2n+1) \log 4 \le (2n+2) \log 4,$$

completing the induction. Part (ii) follows from (i), as $\alpha \log 4 = 4$. //

Corollary 1. $\pi(x) \le C_1 x / \log x$ for $x \ge 2$ and some constant $c_1 \le 3.1 \log 4$.

Proof. By the Theorem and Problems 1. //

Corollary 2. $\psi(x) \le C_1 x$.

Proof. $\psi(x) \le \pi(x) \log x$ and then apply Corollary 1. //

Proposition 2. For m the largest integer with $2^m \le x$, $\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + ... + \theta(x^{1/m})$. //

Proof. See J p. 76.

Proposition 3. (i) $0 \le \psi(x) - \theta(x) \le 6\sqrt{x}$ for $x > 1$.
(ii) $\forall \epsilon > 0, \psi(x) \le (\log 4 + \epsilon)x$ for large enough x.

Proof. For (i), use the result above, and as $\theta(\cdot)$ is increasing:

$$\psi(x) - \theta(x) \leq \theta(\sqrt{x}) + m\theta(x^{1/3}) \qquad (m \leq \log x / \log 2).$$

So by Chebyshev's Upper Estimate for θ, $\psi(x) - \theta(x) \leq x^{1/2} \log 4 + 2x^{1/3} \log x$. But $x^{1/3} \log x \leq \frac{6}{e} x^{1/2}$ (check: the maximum of $\log(x)/x^{\alpha}$ is $1/(\alpha e)$). So $\psi(x) - \theta(x) \leq (\log 4 + 12/e)x^{1/2} < 6x^{1/2}$, giving (i). For (ii), use (i) and the fact that $\theta(x) \leq (\log 4)x$. //

Corollary 3. $(\psi(x) - \theta(x))/x \to 0$ $(x \to \infty)$.

So if either of $\psi(x)/x, \theta(x)/x$ has a limit, both do and they are the same. Now PNT is $\pi(x) \sim li(x) \sim x/\log x$. So ($c = C$ in the first Chebyshev Theorem above) gives:

Theorem (Equivalence Theorem). The following are equivalent:
(i) PNT: $\pi(x) \sim li(x) \sim x/\log x$; (ii) $\psi(x) \sim x$; (iii) $\theta(x) \sim x$.

Cor. 4 (see J. p.77 for proof). $\psi(x) < 2x$ $(x > 1)$.

Powers of primes. Write π^* for the prime-power counting function, $\pi^*(x) := \sum_{p^m \leq x} 1$. Then as above, we find

$$\pi^*(x) = \pi(x) + \pi(\sqrt{x}) + \ldots + \pi(x^{1/m}),$$

with m the largest integer with $2^m \leq x$, and

$$\pi^*(x) - \pi(x) \leq 12C\sqrt{x}/\log x \qquad (x \geq 2),$$

with C s.t. $\pi(x) \leq Cx/\log x$ $(x \geq 2)$. For details, see [J] p.78-79.

Chebyshev's Lower Estimates.
Write $\nu := e_1 - 2e_2$: $\nu(1) = 1$, $\nu(2) = -2$, $\nu(n) = 0$ for $n \geq 2$. Then

$$(u*\nu)(x) = \sum_{i|n} \nu(i).1 = 1 \quad (n \text{ odd } : i = 1 \text{ only }), \quad -1 \quad (n \text{ even } : i = 1, 2).$$

Let $E(x) := \sum_{n \leq x}(u * \nu)(n)$. Then $E(x) = 1$ if $[x]$ is odd, 0 if $[x]$ is even.