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Lecture 21. 28.2.2014.

§5. Newman's theorem

The result of this section gives PNT easily (III.6, L23). It is due to D. J. Newman (1930-2007) in 1980, simplifying work of A. E. Ingham (1900-1967) in 1935. The further simplified version we present is due to J. Korevaar (1923-) in 2004 (Proc. AMS + Nieuw Arch. Wiskunde) (last year we used the Wiener-Ikehara theorem as in Korevaar's book, also of 2004).

Theorem 1. If (i) $f(z) := \sum_1^\infty a_n/n^z$ $(a_n \geq 0)$ converges in $Re\ z > 1$,
(ii) $g(z) := f(z) - A/(z-1)$ has an analytic (so continuous) continuation to $Re\ z \geq 1$, and
(iii) $s_n := \sum_1^n a_k = O(n)$
– then $s_n/n \to A$.

Proof. Put $s(v) := \sum_{n \leq v} a_n$ (so $s(v) = s_n$ on $(n, n+1]$, $s(v) = 0$ on $\mathbb{R}_-$). Then by (iii),

$$s(v)/v = O(1).$$

By partial summation,

$$\begin{aligned} f(z) &:= \sum_1^\infty (s_n - s_{n-1})/n^z = \sum_1^\infty s_n\Big(\frac{1}{n^z} - \frac{1}{(n+1)^z}\Big) \\ &= \sum_1^\infty s_n.z\int_n^{n+1} dv/v^{z+1} = z\int_1^\infty s(v)v^{-z-1}dv. \end{aligned}$$

So by (ii),

$$g(z) - A = f(z) - \frac{A}{z-1} - A = f(z) - \frac{Az}{z-1} = z\int_1^\infty \Big(\frac{s(v)}{v} - A\Big)v^{-z}dv.$$

For $v \geq 1$, write $v = e^t$;

$$\rho(t) := e^{-t}s(e^t) - A = \frac{s(v)}{v} - A.$$

So $\rho(.) = 0$ on $\mathbb{R}_-$ and $\rho(t) = O(1)$.

For $t > u \geq 0$,

$$\rho(t) - \rho(u) = e^{-t}s(e^t) - e^{-u}s(e^u) \geq (e^{-t} - e^{-u})s(e^u)$$

$$\geq -C(1-e^{-(t-u)}) \to 0 \qquad (t,u\to\infty, \quad 0<t-u\to 0),$$

by (iii). In words: ρ is *slowly decreasing (SD)* (i.e., ρ can decrease only slowly – it may well be increasing).

We pass from Dirichlet series to the *Laplace transform (LT)* $L\rho$ of ρ:

$$G(z) := L\rho(z) := \int_0^\infty \rho(t)e^{-zt}dt = \int_1^\infty \Big(\frac{s(v)}{v} - A\Big)v^{-z-1}dv = \frac{g(z+1)-A}{z+1}.$$

By (ii), $G(z) = L\rho(z)$ is analytic in $Re\ z > 0$ and has a continuous extension to $Re\ z \geq 0$. To prove Theorem 1 we have to show

$$\rho(t) \to 0 \qquad (t\to\infty).$$

This follows from the more general Theorem 2 below.

Theorem 2. (i) If $\rho(t) = 0$ on R_- and $|\rho(.)| \leq M$ on $\mathbb{R}_+$, its LT $G := L\rho$ is analytic in $x = Re\ z > 0$.
(ii) If also

$$G(x+iy) \to G(iy): \quad L\rho(x+iy) \to L\rho(iy) \quad (x\downarrow 0) \quad (-R\leq y\leq R)$$

uniformly (or in L_1) – then for all $T, \delta > 0$

$$|\int_T^{T+\delta} \rho(t)dt| \leq \frac{4M}{R} + \frac{1}{2\pi}|\int_{-R}^{R} G(iy).\frac{e^{\delta iy}-1}{y}.\Big(1-\frac{y^2}{R^2}\Big)e^{iTy}dy|.$$

(iii) If R here can be arbitrarily large and ρ is SD, then

$$\rho(t) \to 0 \qquad (T\to\infty).$$

Proof. (i) The integral for $G = L\rho$ converges in $x = Re\ z > 0$, and Laplace transforms are analytic where convergent (we quote this; we know it for Dirichlet series, all we actually need).
(ii) The truncated Laplace transform $G_T(z) := \int_0^T \rho(t)e^{-zt}dt$ is entire. We estimate

$$G_{T+\delta}(0) - G_T(0) = \int_T^{T+\delta} \rho(t)dt. \qquad (*)$$

With $\Gamma := C(0,R)$ the circle centre 0 radius R, by Cauchy's Residue Theorem

$$2\pi i G_T(0) = \int_\Gamma G_T(z).\frac{1}{z}.dz.$$