m3pm16l24.tex.tex

**Lecture 24. 4.3.2014**.

**Theorem (Riemann, 1859)**. The Riemann zeta function satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\frac{1}{2}(1-s))\zeta(1-s). \qquad (FE)$$

*Proof.* We follow Titchmarsh [T], §2.6. From Euler's integral definition of $\Gamma$,

$$\int_0^\infty x^{\frac{1}{2}s-1}e^{-n^2\pi x}dx = \Gamma(\frac{1}{2}s)/n^s\pi^{\frac{1}{2}s} \qquad (\sigma > 0).$$

So for $\sigma > 1$,

$$\begin{aligned}
\Gamma(\frac{1}{2}s)\zeta(s)/\pi^{\frac{1}{2}s} &= \sum_{n=1}^\infty \int_0^\infty x^{\frac{1}{2}s-1}e^{-n^2\pi x}dx \\
&= \int_0^\infty x^{\frac{1}{2}s-1}\sum_{n=1}^\infty e^{-n^2\pi x}dx \qquad \text{(absolute convergence)} \\
&= \int_0^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \qquad \text{(see III.5 for } \Psi).
\end{aligned}$$

Recall ($\theta$ of III.5) $2\Psi(x) + 1 = \frac{1}{\sqrt{x}}(2\Psi(1/x) + 1)$. So

$$\begin{aligned}
\Gamma(\frac{1}{2}s)\zeta(s)/\pi^{\frac{1}{2}s} &= \int_0^1 + \int_1^\infty \ldots \\
&= \int_0^1 x^{\frac{1}{2}s-1}\Big(\frac{1}{\sqrt{x}}\Psi(1/x) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\Big)dx + \int_1^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\Psi(1/x)dx + \int_1^\infty x^{\frac{1}{2}s-1}\Psi(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty (x^{-\frac{1}{2}s-\frac{1}{2}} + x^{\frac{1}{2}s-1})\Psi(x)dx.
\end{aligned}$$

As $\Psi$ decreases exponentially, the integral is convergent for all $s$. So the above holds for *all* $s$ by analytic continuation. Now RHS is invariant under interchanging $s$ and $1-s$, hence so is the LHS, which is $(FE)$. //

**Corollary**. $\zeta(0) = -\frac{1}{2}$; $\zeta(-2n) = 0$ $(n = 1, 2, \ldots)$.

*Proof.* $\Gamma$ has a simple pole at 0 of residue 1; $\zeta$ has a simple pole at 1 of residue 1. So near $s = 0$, $(FE)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ give

$$\frac{2}{s}.\zeta(s) \sim \frac{1}{\sqrt{\pi}}.\Gamma(\frac{1}{2}).(-\frac{1}{s}) = -\frac{1}{s}: \qquad \zeta(0) = -\frac{1}{2}.$$

The RHS of $(FE)$ is holomorphic at $s = 2n$. The LHS contains a (simple) pole from $\Gamma(-\frac{1}{2}s)$, so this must be cancelled by a (simple) zero of $\zeta$: $\zeta(-2n) = 0$. //

The zeros of $\zeta$ at $-2n$ are called the *trivial zeros*.

**Corollary**. The function

$$\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$$

is entire, and satisfies the functional equation

$$\xi(s) = \xi(1-s). \qquad (FE - \xi)$$

*Proof.* The RHS has apparent poles from the $\Gamma$ and $\zeta$ factors. But the pole of the first is cancelled by the factor $1 - s$, and the poles of the second are cancelled by the trivial zeros. So there are no singularities, so $\xi$ is entire. Then $(FE - \xi)$ follows from $(FE)$. //

*Note.* 1. The factor $\frac{1}{2}$ in $\xi$ is for convenience (and historical reasons), and allows us to write

$$\xi(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s + 1)\zeta(s).$$

2. The functional equation shows that $\xi$ is invariant under reflection in the line $\sigma = \frac{1}{2}$ – the *critical line* of III.1 (Riemann, 1859). So we may restrict attention throughout to the half-plane $\sigma \geq \frac{1}{2}$. Since also $\overline{\Gamma(s)} = \Gamma(\overline{s})$, we may restrict attention to $t \geq 0$ – and as (III.4) there is a rectangle $1 - \epsilon \leq \sigma \leq 1, 0 \leq t \leq 2$ on which $\zeta$ is non-zero) to $t \geq 2$.
3. Euler's Reflection Principle $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ also gives reflection-invariance about the same line, and as $(FE)$ shows, there are links between $\Gamma$ and $\zeta$ (cf. WW XII for $\Gamma$, XIII for $\zeta$).
4. For $n = 1, 2, \ldots$, one has ([T], (2.4.3) p.19)

$$\zeta(1-2n) = (-)^n B_n/(2n).$$