m3pm16l25.tex.tex

Lecture 25. 11.3.2014.

IV. PNT WITH REMAINDER

1. Perron's formula.

There is an analogy between power series and Dirichlet series (though this is not exact: a power series has the same circle of convergence and of absolute convergence, while a Dirichlet series may have different half-planes of convergence and absolute convergence). We may recover the coefficients of a power series from the power series by Cauchy's Integral Formulae (M2PM3 II.6, L20). There is an analogous formula for Dirichlet series (Oskar Perron (1880-1975) in 1908). We follow Titchmarsh [T] 3.12, [MV], 5.1, [A], §11.12.

We use the following notation:

$$f(s) := \sum_{1}^{\infty} a_n/n^s$$

is a Dirichlet series, with abscissae of convergence σ_c and of absolute convergence σ_a. We extend $a : n \mapsto a_n$ by setting it to be 0 unless $n \in \mathbb{N}$, and define its *normalised* sum-function by

$$A^*(x) := \sum_{n \leq x} a_n + \frac{1}{2} a_x \qquad (x \geq 0)$$

(this use of 'half the last value if present' is reminiscent of the *Gibbs phenomenon* for Fourier series). The *Heaviside function* is the unit jump-function $H(x) := 0$ on $(-\infty, 0)$, 1 on $[0, \infty)$); we write h for its variant

$$h(x) := 1 \quad (x > 1), \qquad \frac{1}{2} \quad (x = 1) \qquad 0 \quad (0 < x < 1).$$

When convergent, we write $\int_{c-i\infty}^{c+i\infty}$ for $\lim_{T,U\to\infty} \int_{c-iU}^{c+iT}$.

Lemma 1. (ii) For $x > 0$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s ds/s = h(x).$$

(ii)

$$|h(x) - \frac{1}{2\pi i}\int_{c-iU}^{c+iT} x^s ds/s| \leq \frac{x^c}{2\pi|\log x|}\Big(\frac{1}{U}+\frac{1}{T}\Big) \qquad (x \neq 1);$$

$$|h(1) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} ds/s| \leq \frac{c}{\pi T}.$$

Proof. First, take $x > 1$. For $k > c$ a sufficiently large integer, write $b := k - c > 0$, R_c for the rectangle with vertices $c - iU, c + iT, -b + iT, -b - iU$ (draw a diagram!). Now $x^s = e^{s \log x} = 1 + s \log x + O(|s|^2)$ as $s \to 0$, so x^s/s has a simple pole at 0 of residue 1. So by CRT,

$$2\pi i = \Big(\int_{c-iU}^{c+iT} + \int_{c+iT}^{-b+iT} + \int_{-b+iT}^{-b-iU} + \int_{-b-iU}^{c-iU}\Big) x^s \frac{ds}{s} :$$

$$\frac{1}{2\pi i}\int_{c-iU}^{c+iT} x^s ds/s - 1 = \frac{1}{2\pi i}\Big(\int_{-b+iT}^{c+iT} + \int_{-b-iU}^{-b+iT} + \int_{c-iU}^{-b-iU}\Big) x^s ds/s.$$

We estimate the three integrals on the right. Recall $|x^s| = x^\sigma$. In the first integral, $s = \sigma + iT$; $|s| \geq T$, $|1/s| \leq 1/T$)

$$|\int_{-b+iT}^{c+iT} x^s ds/s| \leq \int_{-b}^{c} x^\sigma d\sigma/T| \leq \frac{1}{T}\int_{-\infty}^{c} x^\sigma d\sigma$$

$$= \frac{1}{T}\int_{-\infty}^{c} e^{\sigma \log x} d\sigma = \frac{1}{T}[\frac{e^{\sigma \log x}}{\log x}]_{-\infty}^{c} = \frac{x^c}{T \log x}.$$

In the second, $s = -b + iy$, $-U \leq y \leq T$; $|s| \geq b$, $1/|s| \leq 1/b$:

$$|\int_{-b-iU}^{b+iT} x^s ds/s| \leq |\int_{-U}^{T} x^s \frac{idt}{it}| \leq (T + U)x^{-b}/b.$$

The third is similar to the first, and gives the upper bound $x^c/(U \log x)$.

The case $0 < x < 1$ is similar.

For $x = 1$, the integral may be evaluated explicitly as a $\tan^{-1}$:

$$\int_{c-iT}^{c+iT} ds/s = \int_{-T}^{T} \frac{idy}{c+iy} = \int_{-T}^{T} \frac{y}{c^2+y^2} dy + ic\int_{-T}^{T} \frac{dy}{c^2+y^2} = 2ic\int_{-T}^{T} \frac{dy}{c^2+y^2}$$

(the other integral vanishes: odd integrand, symmetric limits). So

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} ds/s = \frac{c}{\pi}\int_0^T \frac{dy}{c^2+y^2} = \frac{1}{\pi}\arctan(T/c) = \frac{1}{2} - \frac{1}{\pi}\arctan(c/T),$$

giving the result as $\tan\theta > \theta$, $\arctan\theta < \theta$ on $(0, \pi/2)$. //