m3pm16l26.tex
Lecture 26. 14.3.2014.
Cor. For $c, T > 0$,

$$|h(x) - \frac{1}{2\pi i}\int_{c-iU}^{c+iT} x^s ds/s| \leq \frac{x^c}{1 + T|\log x|}.$$

Proof. If $T|\log x| > 1$, this follows from Lemma 1(ii) (as then the 1 in the denominator of the bound can be dropped). Otherwise, we have to show the LHS is $O(x^c)$. Write

$$\int_{c-iT}^{c+iT} x^s ds/s = x^c \int_{c-iT}^{c+iT} ds/s + x^c \int_{c-iT}^{c+iT} (x^{it} - 1) ds/s.$$

The first integral is $O(1)$ (it converges as $T \to \infty$, by Lemma 1(ii)). The second integral is, using the Vinogradov notation $<<$ instead of the Landau O-notation (as $|t/s| = |t/(c + it)| \leq 1$ on the line of integration)

$$\int_{c-iT}^{c+iT} (x^{it} - 1) ds/s << \int_0^T \Big|\frac{x^{it} - 1}{s}\Big| dt = \int_0^T \Big|\frac{e^{it\log x} - 1}{s}\Big| dt \leq \int_0^T \Big|\frac{e^{it\log x} - 1}{t}\Big| dt$$

$$= \int_0^T \log x \frac{|\int_0^t e^{iu\log x} du|}{t} dt \leq \int_0^T \log x dt = T|\log x| \leq 1$$

So both terms are $O(x^c)$, as required. //

Theorem 1 (Perron's formula: first form). For $c \geq \max(0, \sigma_a)$, $T, x \geq 1$,

$$A(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s) x^s ds/s + O(x^c \sum_1^\infty \frac{|a_n|}{n^c(1 + T|\log(x/n)|}).$$

Proof. Replace x by x/n in the Lemma and sum over $n \geq 1$. From the definition of h, $h(x/n)$ is 1 if $n < x$, $\frac{1}{2}$ if $n = x$ and 0 if $n > x$. So the h-term in the Lemma gives $A^*(x)$; the integral term gives $f(s)x^s$; the RHS gives the required error term. //

We shall use the following variant (cf. T, §3.12) in our proof of the PNT with remainder term; it and its proof are *not examinable.* The conditions look artificial, but fit well with our main example, Λ and $-\zeta'/\zeta$.

Theorem 2 (Perron's formula: second form). With $f(s) = \sum_1^\infty a_n/n^s$ a Dirichlet series with abscissa of absolute convergence $\sigma_a < \infty$: if
(i) for some $\alpha \geq 0$,

$$\sum_1^\infty |a_n|/n^\sigma << (\sigma - \sigma_a)^{-a},$$

(ii) for some increasing function M,

$$|a_n| \leq M(n)$$

– then for $x, T \geq 2$, $\sigma \leq \sigma_a$, $c := \sigma_a - \sigma + 1/\log x$,

$$\sum_{n \leq x} a_n/n^s = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)x^w dw/w + O\Big(x^{\sigma_a - \sigma}\frac{(\log x)^a}{T}\Big) + \frac{M(2x)}{x^\sigma}\Big(1 + x\frac{\log T}{T}\Big)\Big).$$

Proof . Here $s = \sigma + i\tau$ is fixed; we use w as the Dirichlet variable in place of s. Apply Theorem 1 to the Dirichlet series $\sum_1^\infty b_n/n^w$ with $b_n := a_n/n^s$. The LHS becomes the LHS above; the Dirichlet series is $\sum_1^\infty b_n/n^w = \sum_1^\infty a_n/n^{s+w} = f(s+w)$. This gives the main term on the RHS above; it suffices to consider the error term.
(i) For $n < \frac{1}{2}x$: $\log(x/n) > \log 2$. So

$$1/(1 + T|log(x/n)|) \leq 1/(1 + T\log 2) << T^{-1}.$$

So the contribution of these n to the error term in Th. 1 is (recalling the above definition of c in terms of x)

$$<< x^c T^{-1} \sum_1^\infty |a_n|/n^{c+\sigma} << x^{\sigma_a - \sigma} T^{-1} (\log x)^a.$$

Similarly for $n > 2\log x$. Combining, this gives the first term in the required error term above.
(ii) For the remaining n, in $[\frac{1}{2}x, 2x]$, we write $n = N + h$ with N the nearest integer to x. Since $\log(1+t)$ is comparable to $|t|$ for t near the origin, $|\log(x/n)| >> |h|/x$ in this range. So the remaining terms contribute

$$x^{-\sigma} \sum_{\frac{1}{2}x \leq n \leq 2x} \frac{|a_n|}{1 + T|\log(x/n)|} << \frac{M(2x)}{x^\sigma}\Big(1 + \sum_{\leq h \leq x/T} 1 + \sum_{x/t < h \leq x+1} \frac{x}{Th}\Big),$$

retaining only the necessary dominant terms. The first sum is $<< x/T$. The second (as the sum approximates an integral) is $<< (x/T).\int_{x/T}^{x+1} du/u << (x/T).\log T$. So we need only retain this term, giving the result. //