m3pm16l27.tex Lecture 27. 14.3.2014.

Our main interest is, of course, the case $a_n = \Lambda(n)$, $f(s) = -\zeta'(s)/\zeta(s)$ relevant to PNT. Recall: $\Lambda(n) \leq \log n$ (II.7 L12: $\Lambda(n) = \log p$ if $n = p^m$, 0 else), and (III.3 L19)

$$-\zeta'(1+\sigma)/\zeta(1+\sigma) << 1/\sigma \qquad (\sigma > 0).$$

So we can apply the result with $M(x) = \log x$, $\sigma_a = 1$, $\sigma = 0$, a = 1 to obtain

$$\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(w)}{\zeta(w)} \frac{x^w}{w} dw + O\left(\frac{x\log x}{T} + \frac{\log(2x)}{x}\left(1 + \frac{x\log T}{T}\right)\right).$$

As $\log(2x) \sim \log x$, the error term is

$$<<\log x\Big(\frac{x}{T} + \frac{1}{x} + \frac{\log T}{T}\Big).$$

This 3-term bracket can be replaced by a simpler 2-term one. We will take $x, T \ge 2$ below, so the 1/x term may (or may not) be small, and can be replaced by "1 +". We then need the larger of x/T and $\log T/T$ when either is large, and this is $\langle x \log T/T$. Combining ("Perron for $-\zeta'/\zeta$ "):

Theorem 3. For $x, T \ge 2$ and $c := 1 + 1/\log x$,

$$\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(w)}{\zeta(w)} \frac{x^w}{w} dw + O\left(\log x \left[1 + \frac{x \log T}{T}\right]\right).$$

This will be a key step in the proof of PNT with remainder. Note. 1. The classical statement of Perron's formula is: for $A(x) := \sum_{n \le x} a_n$, if

$$\alpha(s) := \sum_{1}^{\infty} a_n / n^s (= s \int_1^{\infty} A(x) x^{-s-1} dx) \quad (\sigma > \max(0, \sigma_c)),$$

then for $\sigma_0 > \max(0, \sigma_c)$,

$$A(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) \frac{x^s}{s} ds$$

Passing from A to α in the first formula is a *Mellin transform*; passing from α to A in the second is an *inverse Mellin transform* (Hjalmar Mellin (1854-1933), Finnish mathematician, in 1902). This pair of formulae is analogous

to those for the *Fourier transform* and the *Laplace transform*, to which they are related. There are Stieltjes versions in all three cases.

2. The proof strategy is now clear. The x in PNT (in the forms $\psi(x) \sim x$ or $\psi(x) = x + O(.)$) is the residue of $-\zeta'(s)/\zeta(s).x^s/s$ at s = 1. The above form of Perron's formula suffices for the PNT itself, but a quantitative form such as Theorems 1 or 2 above is needed for PNT with remainder.

2. Further Complex Analysis.

These results will be needed for the proof of PNT with remainder term. *The Gamma function.*

We return to the Gamma function of I.7.

Stirling's formula. Recall that for $n \in \mathbb{N}$ $\Gamma(n+1) = n!$ – the Gamma function is a continuous extension of the factorial. Then (James STIRLING (1692-1770) in 1730)

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}} \qquad (n \to \infty).$$

In terms of the Gamma function,

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} \qquad (x \to \infty).$$

We shall need an estimate for $\Gamma(z)$ with z complex. Recall that Γ has poles at $0, -1, -2, \ldots$ but no zeros, so $1/\Gamma$ is entire (with zeros at $0, -1, -2, \ldots$). For $\delta > 0$, write $D_{\delta} := \{z \in \mathbb{C} : -\pi + \delta < \arg z < \pi - \delta, |z| > 1\}$ (so we can 'go off to infinity' avoiding the poles on the negative real axis). Then

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right) \qquad (z \in D_{\delta}, \ |z| \to \infty)$$

(the RHS is an asymptotic expansion). This yields an asymptotic expansion for $\log \Gamma(z)$ (involving the Bernoulli numbers – see e.g. WW, 12.33), and hence (all we shall need)

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O_{\delta}(1/|z|) \qquad (z \in D_{\delta}).$$
 (St)

It can be shown that the error term here has derivative $O_{\delta}(1/|z|^2)$ (as one would expect). So differentiating, the error term is negligible, and one obtains the *complex Stirling formula*

$$\Gamma'(z)/\Gamma(z) = \log z + O_{\delta}(1/|z|) \qquad (z \in D_{\delta}).$$
(St)

This logarithm occurs again in the zero-free region for $\zeta(s)$ (IV.3), and the bound for $-\zeta'/\zeta$ (IV.4), and hence in our error term in PNT (IV.5).