m3pm16l33.tex
Lecture 33. 28.3.2014.

6. Equivalents of PNT

Recall the Equivalence Theorem (II.2 L17): the following are elementarily equivalent:

PNT: $\pi(x) \sim li(x) \sim x/\log x$; $\psi(x) \sim x$; $\theta(x) \sim x$.

One can extend these equivalences.

Theorem. The following are elementarily equivalent:
(i) $\psi(x) \sim x$;
(ii) $M(x) := \sum_{n\leq x} \mu(n) = o(x)$;
(iii) $\sum_1^\infty \mu(n)/n = 0$ (i.e., the series converges to 0);
(iv) $\sum_{n\leq x} \Lambda(n)/n = \log x + \gamma + o(1)$.

Proof. (iii) $\Rightarrow$ (ii). If (iii) holds,

$$m(x) := \sum_{n\leq x} \mu(n)/n = \int_1^x dM(u)/u = o(1) :$$

$$M(x) := \sum_{n\leq x} \mu(n) = \int_1^x u dm(u) = xm(x) - \int_1^x m(u)du = o(x).$$

(ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (iii). See [MV] §8.1.
(iv) $\Rightarrow$ (i). Write

$$f(x) := \sum_{n\leq x} \Lambda(n)/n = \int_1^x d\psi(u)/u :$$

$$\frac{1}{x} f(u)du = \frac{1}{x}\int_1^x du \int_1^u \frac{d\psi(v)}{v} = \frac{1}{x}\int_1^x \frac{d\psi(v)}{v}\int_v^x du = \frac{1}{x}\int_1^x (x-v)\frac{d\psi(v)}{v}$$
$$= \int_1^x \frac{d\psi(v)}{v} - \frac{1}{x}\int_1^x d\psi(v) = f(x) - \frac{\psi(x)-1}{x} :$$
$$\frac{\psi(x)-1}{x} = f(x) - \frac{1}{x}\int_1^x f(u)du.$$

By (iv), $f(x) = \log x - \gamma + o(1)$. So

$$\frac{1}{x}\int_1^x f(u)du = \frac{1}{x}\int_1^x \log u du - \frac{(x-1)}{x}.\gamma + o(1) = \frac{1}{x}[u\log u - u]_1^x - \gamma + o(1) = \log x - 1 - \gamma + o(1).$$

Subtract:

$$f(x) - \frac{1}{x}\int_1^x f(u)du = 1 + o(1): \qquad \frac{\psi(x)}{x} = 1 + o(1): \qquad \psi(x) \sim x,$$

which is (i).
(i) $\Rightarrow$ (iv): This uses DHI plus the arguments in MV cited above; omitted. //

One can combine the results above ([MV], §6.2):

Theorem. For some absolute constant $c > 0$,

$$\psi(x) = x + O(x \exp\{-c\sqrt{\log x}\}),$$
$$\theta(x) = x + O(x \exp\{-c\sqrt{\log x}\}),$$
$$\pi(x) = li(x) + O(x \exp\{-c\sqrt{\log x}\}),$$
$$M(x) := \sum_{n \le x} \mu(n) = O(x \exp\{-c\sqrt{\log x}\}),$$
$$m(x) := \sum_{n \le x} \mu(n)/n = O(\exp\{-c\sqrt{\log x}\}).$$

Similarly, with λ the Liouville function and Q the square-free counting function,

$$\sum_{n \le x} \lambda(n) = O(x \exp\{-c\sqrt{\log x}\}), \quad Q(x) = \frac{6}{\pi^2}x + O(x \exp\{-\frac{1}{2}c\sqrt{\log x}\}).$$

The Riemann Hypothesis, (RH).
Recall (RH) (III.1, L15), that all the (non-trivial) zeros $\beta + i\gamma$ of ζ have $\beta = \frac{1}{2}$. We close with two results linking (RH) to PNT with remainder and (ZFR):
(i) Let θ be the lower bound of those real numbers ξ with

$$\psi(x) = x + O(x^{\xi}).$$

Then

$$\theta = \sup_{\rho} \beta$$

(the supremum is over all (non-trivial) zeros $\rho = \beta + i\gamma$ of ζ).
(ii) (RH) is equivalent to

$$\forall \epsilon > 0, \qquad \psi(x) = x + O_{\epsilon}(x^{\frac{1}{2}+\epsilon}).$$