M3PM16/M4PM16 MASTERY QUESTION 2014

$Q(x)$ and PNT.

State without proof Dirichlet's hyperbola identity.

Write $Q(x)$ for the number of *square-free* natural numbers $n \leq x$; thus $Q(x) = \sum_{n \leq x} |\mu|(n)$, with μ the Möbius function. You may quote that $|\mu| = \nu * u$, where $u(n) \equiv 1$ and $\nu(n) := \mu(d)$ if $n = d^2$ is a square, 0 otherwise.

Given the Prime Number Theorem in the form

$$M(x) := \sum_{n \leq x} \mu(n) = o(x),$$

show that

$$Q(x) = \frac{6}{\pi^2}x + o(\sqrt{x}).$$

(You may find it helpful to use Dirichlet's hyperbola identity with $a = u$, $b = \nu$, letting first $x \to \infty$ and then $y \to \infty$. You may quote that $\zeta(2) = \pi^2/6$.)

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