M4PM16 SOLUTION TO MASTERY QUESTION, 2013

THEOREM (Chebyshev). If

$$\ell := \liminf \frac{\pi(x)}{x/\log x}, \qquad L := \limsup \frac{\pi(x)}{x/\log x},$$

then

$$\ell \le 1 \le L.$$

In particular, if the limit exists, it is 1 (as in PNT).

Proof. For all $\epsilon > 0$ there exists x_0 such that for $x \ge x_0$

$$\ell - \epsilon \le \frac{\pi(x)}{x/\log x}, \qquad \frac{\pi(x)}{x/\log x} \le L + \epsilon. \qquad [\mathbf{4}], \text{unseen}$$

For the lower bound, partial summation gives, as $0 < \pi(u) \le u$,

$$\sum_{p \le x} \frac{1}{p} \ge \sum_{x_0 < p \le x} \frac{1}{p} = \frac{\pi(x)}{x} - \frac{\pi(x_0)}{x_0} + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt,$$

$$\ge -1 + \int_{x_0}^{x} \frac{\pi(t)}{t^2} dt \ge -1 + (\ell - \epsilon) \int_{x_0}^{x} \frac{dt}{t \log t} \ge (\ell - \epsilon) \log\log x + O_\epsilon(1)$$

($\int_1^x dt/(t \log t) = \int_1^x d\log t/\log t = \log\log x$). [**6**], unseen

But by Mertens' Second Theorem,

$$\sum_{p \le x} 1/p = \log\log x + c_1 + O(1/\log x). \qquad [\mathbf{2}], \text{seen}$$

Combining,

$$\log\log x + c_1 + O(1/\log x) \ge (\ell - \epsilon) \log\log x + O_\epsilon(1). \qquad [\mathbf{2}], \text{unseen}$$

In particular,

$$1 \ge \ell - \epsilon.$$

This holds for all $\epsilon > 0$. So $\ell \le 1$. [**3**], unseen

The upper bound is similar but slightly simpler: $1 \le L$. // [**3**], unseen

Note. Chebyshev's Theorem can also be stated as

$$\liminf \pi(.)/li(.) \le 1 \le \pi(.)/li(.).$$

[Proof unseen. The result was stated and discussed in lectures; Mertens' Second Theorem was proved in lectures,]

NHB