M3PM16/M4PM16 MASTERY QUESTION SOLUTION 2014

Theorem (Dirichlet's Hyperbola identity, DHI). For $A(x) := \sum_{n\le x} a_n$, $B(x) := \sum_{n\le x} b_n$,$1 < y < x$,

$$\sum_{n\le x}(a*b)(n) = \sum_{j\le y} a(j)B(x/j) + \sum_{k\le x/y} b(k)A(x/k) - A(y)B(x/y). \qquad [\mathbf{2}]$$

With $\nu(n) := \mu(d)$ if $n = d^2$, 0 otherwise, given $|\mu| = \nu * u$.
In DHI (II.9 L14), take $a = u$, sum-function $A = [.]$, $b = \nu$, sum-function

$$B(x) := \sum_{n\le x} \nu(n) = \sum_{d^2\le x} \mu(d) = \sum_{d\le\sqrt{x}} \mu(d) = M(\sqrt{x}).$$

So DHI gives, for $1 < y < x$,

$$\begin{aligned} Q(x) &= \sum_{n\le x} |\mu|(n) = \sum_{n\le x}(u*\nu)(x) \\ &= \sum_{j\le y} M(\sqrt{x/j}) + \sum_{k\le x/y} \nu(k)[x/k] - [y]M(\sqrt{x/y}) \end{aligned}$$

$$= \sum_{j\le y} M(\sqrt{x/j}) + \sum_{d\le\sqrt{x/y}} \mu(d)[x/d^2] - [y]M(\sqrt{x/y}). \qquad (i) \quad [\mathbf{5}]$$

Writing $[.] = . + \{.\} = . + O(1)$, the second term in (i) is

$$\sum_{d\le\sqrt{x/y}} \mu(d)(x/d^2 + O(1)) = x \sum_{d\le\sqrt{x/y}} \mu(d)/d^2 + O(\sqrt{x/y})$$

$$= x(\sum_1^\infty \mu(d)/d^2 - \sum_{d>\sqrt{x/y}} ...) + O(\sqrt{x/y}) = \frac{6}{\pi^2}x - x\int_{\sqrt{x/y}} dM(u)/u^2 + O(\sqrt{x/y}).$$

$$(ii) \quad [\mathbf{5}]$$

But as $M(x) = o(x)$,

$$\int_z^\infty dM(u)/u^2 = -M(z)/z^2 + 2\int_z^\infty M(u)du/u^3 = o(1/z) + \int_z^\infty o(1/u^2).du = o(1/z).$$

So the second term in (ii) is $x.o(\sqrt{y/x}) = o(\sqrt{xy}) = o_y(\sqrt{x})$. Similarly, $\sum_{j\le y} M(\sqrt{x/j}) = o_y(\sqrt{x})$. Combining,

$$Q(x) = \frac{6}{\pi^2}x + o_y(x) + O(\sqrt{x/y}). \qquad [\mathbf{4}]$$

Take the first (x-) term on the right to the left, divide by $\sqrt{x}$, and let $x \to \infty$. The right becomes $o_y(1) + O(1/\sqrt{y})$. We can then let $y \to \infty$, to get

$$\limsup_{x\to\infty} |Q(x) - \frac{6}{\pi^2}x|/\sqrt{x} = 0 : \qquad Q(x) = \frac{6}{\pi^2}x + o(\sqrt{x}). \qquad // \quad [\mathbf{4}]$$

[Seen – Problems] NHB