SOLUTIONS TO MOCK EXAMINATION 2012

Q1 (Prob5 Q1,2). (i) $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. So

$$0 < -\log(1-1/p) = \frac{1}{2p^2} + \frac{1}{3p^3} + \ldots < \frac{1}{2p^2} + \frac{1}{2p^3} + \ldots = \frac{1}{2p(p-1)},$$

summing the GP. Also

$$\sum_p \frac{1}{p(p-1)} < \sum_n \frac{1}{n(n-1)} < \infty.$$

So by the Comparison Text,

$$\sum_p \{-\log(1-1/p) - 1/p\} \text{ converges.}$$

But (Euler, II.4) $\sum 1/p$ diverges. So $\sum\{-\log(1-1/p)\}$ diverges also. That is, the infinite product $\prod(1-1/p)$ diverges to 0 (I.5).

(ii). With $N(x,r)$ the number of $n \le x$ not divisible by any of the first r primes p_k, then

$$\pi(x) \le N(x,r) + r$$

(a prime $p \le x$ is either one of the first r or not divisible by any of the first r). By Inclusion-Exclusion (Problems 4 Q2),

$$N(x,r) = [x] - \sum_i [x/p_i] + \sum_{ij} [x/p_i p_j] \ldots$$

The number of square brackets is

$$1 + \binom{r}{1} + \binom{r}{2} + \ldots = (1+1)^r = 2^r.$$

Replacing each [.] by . introduces an error of < 1, so

$$N(x,r) < x - \sum_i x/p_i + \sum_{ij} x/p_i p_j \ldots + 2^r = x \prod_1^r (1 - 1/p_k) + 2^r.$$

Combining,

$$\pi(x) \le x \prod_1^r (1-1/p_k) + 2^r + r : \qquad \pi(x)/x \le \prod_1^r (1-1/p_k) + (2^r + r)/x.$$

As the product diverges (Q1), $\prod_1^r$ can be made arbitrarily small by taking r large enough. Then letting $x \to \infty$ gives $\pi(x)/x \to 0$. //

Q2 (Prob2 Q2-4). (i) Integrating by parts,

$$li(x) = \int_2^x \frac{dt}{\log t} = \Big[\frac{x}{\log x}\Big]_2^x - \int_2^x \frac{t.(-)}{\log^2 t}.\frac{1}{t}dt = const + \frac{x}{\log x} + \int_2^x \frac{du}{\log^2 u}.$$

For $x \geq 4$,

$$\int_2^x \frac{du}{\log^2 u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \int_2^{\sqrt{x}} \frac{du}{\log^2 2} + \int_{\sqrt{x}}^x \frac{du}{\log^2(\sqrt{x})} = \frac{\sqrt{x}-2}{\log^2 2} + \frac{x-\sqrt{x}}{\log^2\sqrt{x}}.$$

The dominant term is the first part of the second term, which is $O(x/\log^2 x)$.
(ii). Taking $x = p_n$ in $\pi(x) := \sum_{p\leq x} 1$ gives $\pi(p_n) = \sum_{p\leq p_n} 1 = n$.
By PNT, $\pi(x) \sim x/\log x$, so $n \sim p_n/\log p_n$:

$$\frac{n \log p_n}{p_n} \to 1. \tag{1}$$

Taking logs of (1), $\log n + \log\log p_n - \log p_n \to 0$. Dividing this by $\log p_n$,

$$\frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \to 0.$$

But $\log x = o(x)$, so $\log\log p_n = o(\log p_n)$, so this says

$$\frac{\log n}{\log p_n} \to 1. \tag{2}$$

Multiply (1) and (2): $n \log n/\log p_n \to 1$, i.e. $p_n \sim n \log n$. //
(iii). By PNT and (i), $\pi(x) = x/\log x + O(x/\log^2 x)$. So taking $x = p_n$,

$$n = \frac{p_n}{\log p_n} + O(\frac{p_n}{\log^2 p_n}), \quad = \frac{p_n}{\log p_n} + O(\frac{n \log n}{\log^2 n}),$$

using (ii) and $p_n \geq n$. So

$$p_n = n(1 + O(1/\log n))\ \log p_n. \tag{3}$$

By (i),

$$\log p_n = \log n + \log\log n + o(1). \tag{4}$$

Substituting (4) in (3) gives the result. //

Q3 (Prob6 Q2). (i)

$$\log \sin z = \log z + \sum_1^\infty \log(1 - \frac{z^2}{n^2\pi^2}),$$

$$\cot z = 1/z - \sum_1^\infty \frac{\frac{2z}{n^2\pi^2}}{1 - \frac{z^2}{n^2\pi^2}}.$$

Multiplying by z and expanding the geometric series,

$$z \cot z = 1 - 2\sum_{n=1}^\infty \sum_{k=1}^\infty (z/n\pi)^{2k}. \tag{1}$$

As $\sum_1^\infty 1/n^{2k} = \zeta(2k)$,

$$z \cot z = 1 - 2\sum_{k=1}^\infty z^{2k}\zeta(2k)/\pi^{2k}.$$

(ii)

$$\cot z = \cos z/\sin z = \frac{1}{2}(e^{iz} - e^{-iz})/\frac{1}{2i}(e^{iz} - e^{-iz}) = i\frac{e^{2iz}+1}{e^{2iz}-1} = i + \frac{2i}{e^{2iz}-1}.$$

So

$$z\cot z = iz + \frac{2iz}{e^{2iz}-1} = iz + 1 - iz + \sum_2^\infty (2iz)^n B_n/n! = 1 + \sum_1^\infty B_{2k}(-)^k \frac{2^{2k}z^{2k}}{(2k)!}. \tag{2}$$

Equating coefficients of z^{2k} in (1), (2): For $k = 1, 2, \ldots$,

$$-2\zeta(2k)/(\pi^{2k} = (-)^k B_{2k} 2^{2k}/(2k)! : \qquad \zeta(2k) = (-)^{k+1}(2\pi)^{2k}B_{2k}/(2(2k)!).$$

Check (not required for the question!). Taking $n = 1, 2, 3$ and using $B_2 = 1/6$, $B_4 = -1/30$ and $B_6 = 1/42$ gives $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$.

Q4. (i) (Lectures, II.7). (i) Mertens' Second Theorem: $\sum_{p \le x} 1/p = \log\log x + C_1 + O(1/\log x)$ for some constant C_1.
(ii) (Problems 7 Q1). Mertens' Second Theorem for prime powers:

$$\sum_{p^n \le x} 1/p^n = \log\log x + C_2 + O(1/\log x), \qquad C_2 := C_1 + S, \qquad S := \sum_p \frac{1}{p(p-1)}.$$

Proof. Write $q := p^n$ for a generic prime power, and for primes p with $p^2 \le x$, let r_p be the largest 'relevant power' (largest r with $p^r \le x$). Then

$$\Delta := \sum_{q \le x} 1/q - \sum_{p \le x} 1/p = \sum_{p \le \sqrt{x}} \sum_{r=2}^{r_p} 1/p^r.$$

But $\sum_2^\infty 1/p_r = 1/(p(p-1))$, summing the GP, so

$$\Delta \le \sum_p \frac{1}{p(p-1)} = S$$

(above). Write

$$S_0 := \sum_{p \le \sqrt{x}} \frac{1}{p(p-1)};$$

then

$$\begin{aligned} S - S_0 \le \sum_{p > \sqrt{x}} \quad &< \quad \sum_{n < \sqrt{x}} \frac{1}{n(n-1)} \\ &= \quad \frac{1}{\sqrt{x}} \qquad \left(\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \text{ sum telescopes}\right) \\ &\le \quad 2/\sqrt{x}. \end{aligned}$$

As $p^{r_p+1} \ge x$:

$$\sum_{r > r_p} \frac{1}{p^r} < \frac{1}{x}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \frac{1}{x(1-1/p)} \le 2/\sqrt{x} \qquad (p \ge 2).$$

So

$$S_0 - \Delta = \sum_{p \le \sqrt{x}} \sum_{r > r_p} 1/p^r < \pi(\sqrt{x}).2/x \le 2/\sqrt{x}$$

($\pi(x) := \sum_{p \le x} 1 \le \sum_{n \le x} 1 \le x$). Combining, $S - \Delta \le 4/\sqrt{x} = O(1/\log x)$. So the difference Δ in the sums here and in Mertens' Second Theorem is $S + O(1/\log x)$, and the result follows from Mertens' Second Theorem. //

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