m3pm16prob3.tex

M3PM16/M4PM16 PROBLEMS 3. 31.1.2014

Q1. Prove Euclid's theorem that there are infinitely many primes directly – i.e., without proof by contradiction.

Q2. (i) If p_n is the nth prime, show (by induction or otherwise) that $p_n \le 2^{2^n}$.
(ii) Deduce that

$$\pi(x) := \sum_{p \le x} 1 \ge \log \log x.$$

Q3. (i) If $2, 3, \ldots, p_j$ are the first j primes and $N(x)$ is the number of $n \le x$ not divisible by any prime $p > p_j$, show that

$$N(x) \le 2^j \sqrt{x}.$$

(ii) Hence or otherwise show that $\sum_p 1/p$ diverges.

Q4. Improving Q3, show that (i) $\pi(x) \ge \log x$,
(ii) $p_n \le 4^n$.

Q5. Show that there are arbitrarily long gaps between primes.

Q6. Show that there are infinitely may primes of the form $p = 4n + 3$.

Note. 1. These questions exploit the idea in Euclid's proof, and push it further.
2. You may find HW Ch. 1, 2 and J Ch. 1 helpful – and I would encourage you to read them anyway.
3. Technically, these results belong to Elementary Number Theory (ENT) rather than Analytic number Theory (ANT) – but it would be pedantic and silly to exclude them on that account. This is good mathematics, of interest in its own right and highly relevant to later results in the course.

I am a Minimalist at heart (as are mathematicians generally), and the Minimalist's motto is *less is more*. The tension between ENT and ANT is between strategic minimalism (less in the preliminaries) and tactical minimalism (less in the length of the current proof). Of course, in ANT one can prove more – and there 'more is more'. NHB