

M3PM16/M4PM16 PROBLEMS 6. 21.2.2014

Q1. Show (by using Chebyshev's Upper Estimate, or otherwise) that

$$\sum \frac{1}{p \log p} < \infty.$$

Q2. Show that the following are equivalent:

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \quad (\pi)$$

$$\theta(x) = x + O\left(\frac{x}{\log x}\right), \quad (\theta)$$

$$\psi(x) = x + O\left(\frac{x}{\log x}\right). \quad (\psi)$$

Q3 ($\zeta(2n)$ and the Bernoulli numbers). Define the *Bernoulli numbers* B_n by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$$

(one can check that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$, ... ; also all $B_{2n+1} = 0$ and all B_{2n} are rational).

(i) Show that

$$z \cot z = 1 + \sum_{j=2}^{\infty} B_j (2ix)^j / j!$$

(ii) From the Weierstrass product for \sin , $\sin z = z \prod_{n=1}^{\infty} (1 - z^2/n^2\pi^2)$ ([T] §3.2 p.114, or M2PM3, Handout), show by taking logs and differentiating that

$$z \cot z = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (z/n\pi)^{2k} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) (z/\pi)^{2k}.$$

(ii) Deduce Euler's formula

$$\zeta(2n) = (-1)^{n+1} (2\pi)^{2n} B_{2n} / 2(2n!);$$

so $\zeta(2) = \pi^2/6$ (again!), $\zeta(4) = \pi^4/90$ (again!), $\zeta(6) = \pi^6/945$, and $\zeta(2n)$ is a rational multiple of π^{2n} (so is irrational, indeed transcendental). NHB