m3pm16prob6.tex

M3PM16/M4PM16 PROBLEMS 7. 28.2.2014

Q1 *Prime divisor functions.* The prime divisor functions Ω, ω count the prime divisors with and without multiplicity: if $n = p_1^{r_1} \dots p_k^{r_k}$,

$$\Omega(n) := r_1 + \dots + r_k, \qquad \omega(n) := k.$$

Show that $(-1)^\Omega$ is completely multiplicative and $(-1)^\omega$ (note: $\mu = (-)^\omega$) is multiplicative.

Q2 *The Liouville function* λ. This is defined by $\lambda := (-)^\Omega$:

$$\lambda = (-)^\Omega, \qquad \mu = (-)^\omega$$

(so $\lambda(p) = -1$, and λ is completely multiplicative, by Q1). Show that:
(i) λ has Dirichlet series

$$\sum_{n=1}^{\infty} \lambda(n)/n^s = \zeta(2s)/\zeta(s);$$

(ii) λ is the convolution inverse of $|\mu|$:

$$\lambda * |\mu| = e.$$

Q3. Define ν by $\nu(n) := \mu(d)$ if $n = d^2$, 0 otherwise. Show that (with $u \equiv 1$)

$$|\mu| = \nu * u.$$

Q4 *The number of square-free integers.*
With $Q(x) := \sum_{n \leq x} |\mu(n)|$ the number of square-free (quadratfrei, whence Q) integers $n \leq x$, show that :
(i) $[x] = \sum_{m \leq \sqrt{x}} Q(x/d^2)$;
(ii) $Q(x) = \sum_{m \leq \sqrt{x}} \mu(m)[x/m^2]$.
(iii)

$$Q(x) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

NHB