m3pm16soln1.tex

M3PM16/M4PM16 SOLUTIONS 1. 24.1.2014

Q1. (i) If x is a terminating decimal, x is a rational of the form $n + m/10^k$.
If x is a recurring decimal, say

$$x = n.a_1 \dots a_k b_1 \dots b_\ell \dots b_1 \dots b_\ell \dots ,$$

x is $n.a_1 \dots a_k$ (rational, above) $+y$, where writing

$$b := b_1/10 + \dots b_\ell/10^\ell$$

(rational, above), y is a geometric series with first term $b/10^k$ and common ratio $10^{-\ell}$. So

$$y = b.10^{-k}/(1 - 10^{-\ell}),$$

rational, so x is rational.

If x is rational, $x = m/n$ say:
(a) take off its integer part – so reducing to $0 \leq m < n$,
(b) cancel m/n down to its lowest terms.
(ii) Now find the decimal expansion of m/n by the Long Division Algorithm. Let the remainders obtained by $r_1, r_2, \dots$. The expansion *terminates* if some $r_k = 0$. It *recurs* if some remainder has *already occurred*. As there are only $n - 1$ different possible non-zero remainders, the expansion must terminate (with remainder 0) or recur (with a remainder the first repeat of one of $1, 2, \dots, n - 1$) after at most $n - 1$ places.
(The examples 1/7, 2/7, 3/7, 4/7, 5/7, 6/7 show that all $n - 1$ places may be needed.)
(iii) Similarly with 10 replaced by 2,3, ...

Q2 ([L], 12-13).

$$li(x) := \int_2^x \frac{du}{\log u} = \Big[\frac{u}{\log u}\Big]_2^x - 2\int_2^x d\Big(\frac{1}{\log u}\Big) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{du}{\log^2 u}.$$

For $x \geq 4$,

$$0 < \int_2^x \frac{du}{\log^2 u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \int_2^{\sqrt{x}} \frac{2}{\log^2 2} + \int_{\sqrt{x}}^x \frac{du}{\log^2 x}$$

$$= \frac{\sqrt{x}-2}{\log^2 2} + \frac{x-\sqrt{x}}{\frac{1}{4}\log^2 x} < \frac{\sqrt{x}}{\log^2 2} + \frac{4x}{\log^2 x} = o\Big(\frac{x}{\log x}\Big).$$

The LH inequality gives

$$\liminf li(x)/\frac{x}{\log x} \geq 1.$$

The RH inequality gives

$$\limsup li(x)/\frac{x}{\log x} \leq 1.$$

Combining,

$$li(x)/\frac{x}{\log x} \to 1 : \qquad li(x) \sim \frac{x}{\log x}.$$

Q3 ([L], 13-14). Integrating by parts $m+1$ times,

$$li(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{m!x}{\log^{m+1} x} + const + (m+1)! \int_2^x \frac{du}{\log^{m+2} u}.$$

For $x \geq 4$, as before,

$$0 < \int_2^x \frac{du}{\log^{m+2} u} = \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x < \int_2^{\sqrt{x}} \frac{du}{\log^{m+2} 2} + \int_{\sqrt{x}}^x \frac{du}{\log^{m+2}(\sqrt{x})}$$

$$< \frac{\sqrt{x}-2}{\log^{m+2} 2} + \frac{x-\sqrt{x}}{2^{-m-2}\log^{m+2} x} = o\Big(\frac{x}{\log^{m+1} x}\Big).$$

So

$$li(x) - \Big(\frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{(m-1)!x}{\log^m x}\Big) = \frac{m!x}{\log^{m+1} x}(1 + o(1)). \qquad //$$

Note. Numerical evidence shows that $li(x)$ gives a much better approximation than $x/\log\, x$, in line with Q1, Q2, and we shall prefer it – particularly in PNT with any error term – see Problems 2.

Q4 ([L], 214-5). Taking $x = p_n$ in $\pi(x) := \sum_{p \leq x} 1$ gives

$$\pi(p_n) = \sum_{p \leq p_n} 1 = n.$$

By PNT, $\pi(x) \sim x/\log x$, so $n \sim p_n/\log p_n$:

$$\frac{n \log p_n}{p_n} \to 1. \tag{i}$$

Taking logs of (i), $\log n + \log\log p_n - \log p_n \to 0$. Dividing this by $\log p_n$,

$$\frac{\log n}{\log p_n} + \frac{\log\log p_n}{\log p_n} - 1 \to 0.$$

But $\log x = o(x)$, so $\log\log p_n = o(\log p_n)$, so this says

$$\frac{\log n}{\log p_n} \to 1. \tag{ii}$$

Multiply (i) and (ii): $n \log n/\log p_n \to 1$, i.e. $p_n \sim n \log n$. //

NHB