m3pm16soln3.tex

M3PM16/M4PM16 SOLUTIONS 3. 7.2.2014

Q1 (HW §2.1). Let $p_1, \ldots, p_n$ be the first n primes. Then

$$q := 1 + p_1 p_2 \ldots p_n$$

is not divisible by any p_i $(i = 1, \ldots, n)$ – it has remainder 1. So by FTA, either q is itself prime, or it has a prime factor, which by above is not in the list $p_1, \ldots, p_n$. Either way, there is a new prime, p_{n+1}. As this holds for every $n = 1, 2, \ldots$, there are infinitely many primes – i.e., $\pi(x) \to \infty$ as $x \to \infty$.

Q2 (HW §2.2). From the proof of Q1,

$$p_{n+1} < p_n^n + 1. \qquad (*)$$

Suppose inductively that $p_n < 2^{2^n}$ for $n = 1, \ldots, N$. Then

$$p_{N+1} \leq p_1 \ldots p_N + 1 < 2^{2+4+\ldots+2^N} + 1 = 2^{2^{N+1}-2} + 1 = \frac{1}{4} 2^{2^{N+1}} + 1 < 2^{2^{N+1}},$$

completing the induction.
(ii) For $n \geq 4$, let $x \in (e^{e^{n-1}}, e^{e^n}]$. Then $e^{n-1} > 2^n$ [we need $n \geq 4$ here: $e^2 < 8 = 2^3$] [$n - 1 > n \log 2$: $0.75 = 1 - \frac{1}{4} > \log 2 = 0.693..$]. So

$$e^{e^{n-1}} > e^{2^n} > 2^{2^n}.$$

So

$$\pi(x) \geq \pi(e^{e^{n-1}} \geq \pi(2^{2^n}) \geq n,$$

by $(*)$. But $\log \log x \leq n$ (as $x \leq e^{e^n}$). Combining,

$$\pi(x) \geq \log \log x$$

for e^{e^3}. But $\pi(x)$ reaches the level 3 when $x = 5$, while $\log \log x$ does so only for $x = e^{e^3}$. So the inequality holds also for $x \leq e^{e^3}$, so for all x. //

Q2 (HW §2.6, Th. 20). (i) If $2, 3, \ldots, p_j$ are the first j primes and N is the number of $n \leq x$ not divisible by any $p > p_j$: each such n is of the form

$$n = n_1^2 m, \qquad m = p_1^{c_1} \ldots p_j^{c_j}, \qquad c_i = 0 \text{ or } 1$$

(any even powers of p_i being absorbed in n_1^2). There are 2^j choices of the powers c_i, so $\#m = 2^j$. Also $n_1 \leq \sqrt{n} \leq \sqrt{x}$, so $\#n_1 \leq \sqrt{x}$. Combining,

$$N(x) = \#n \leq \#m.\#n_1 = 2^j\sqrt{x} : \qquad N(x) \leq 2^j\sqrt{x}.$$

(ii) If $\sum 1/p < \infty$: choose j so large that $\sum_{j+1}^{\infty} 1/p_k < 1/2$. The number of $n \leq x$ divisible by p is $[x/p] \leq x/p$. So the number of $n \leq x$ divisible by at least one of the p_k $(k \geq j+1)$ is $\leq x\sum_{j+1}^{\infty} 1/p_k < x/2$. Combining with (i):

$$\frac{1}{2}x < N(x) \leq 2^j\sqrt{x} : \qquad \sqrt{x} \leq 2^{j+1} : \qquad x \leq 2^{2j+2}.$$

This is false for large enough x (j is fixed). So $\sum 1/p$ diverges.
(iii) Take $j = \pi(x)$. So $p_{j+1} > x$, and $N(x) = x$. Then (i) gives

$$x = N(x) \leq 2^{\pi(x)}\sqrt{x} : \qquad 2^{\pi(x)} \geq \sqrt{x} : \qquad \pi(x) \geq \frac{\log x}{2\log 2}.$$

(iv) Taking $x = p_n$: $\pi(x) = n$: $2^n \geq \sqrt{p_n}$, so $p_n \leq 4^n$.

Q3 (HW Th. 5). If $2, 3, \ldots, p$ are all the primes up to p, then all numbers n up to p are divisible by at least one of these primes, p' say, by FTA: $p'|n$, $n = p'r'$ say. So if $q := 2.3.5\ldots p = \prod p'$, then all the $p-1$ numbers $q+2, q+3, q+4, \ldots, q+p$ are composite: for each is of the form

$$q + n = q + p'r' = p'.\prod p'' + p'r' = p'(r' + \prod p''),$$

where the product is over the primes other than p in the prime-power factorisation of n (counted with multiplicity), and any repetitions of p. So this string of $p-1$ consecutive numbers forms (part of) a gap between primes. There are arbitrarily large p (Euclid), so arbitrarily long gaps between primes.

Q4 (HW, Th. 11). Again as in Euclid, write $q := 2^2.3, \ldots p - 1$. Then $4|q$, so $q = 4n+3$ for some n (residue $3 = -1 \bmod 4$), and q is not divisible by any of the primes up to p. It cannot be a product of primes of the form $4n+1$, or it too would be of this form. So q is of the form $4n+3$, and there are infinitely may such q, one for each p.
Note. There are also infinitely many primes of the form $4n+1$, but this is harder (HW, Th. 14). More is true: from Dirichlet's PNT for primes in AP (IV.5), 'half the primes are $4n+1$, half are $4n+3$'.

NHB