

**M3PM16/M4PM16 SOLUTIONS 5. 21.2.2014**

Q1.  $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$ . So

$$0 < -\log(1-1/p) = \frac{1}{2p^2} + \frac{1}{3p^3} + \dots < \frac{1}{2p^2} + \frac{1}{2p^3} + \dots = \frac{1}{2p(p-1)},$$

summing the GP. Also

$$\sum_p \frac{1}{p(p-1)} < \sum_n \frac{1}{n(n-1)} < \infty.$$

So by the Comparison Text,

$$\sum_p \{-\log(1-1/p) - 1/p\} \text{ converges.}$$

But (Euler, II.4)  $\sum 1/p$  diverges. So  $\sum \{-\log(1-1/p)\}$  diverges also. That is, the infinite product  $\prod(1-1/p)$  diverges to 0 (I.5).

Q2 (HW, 4th ed., §22.7 – I find this proof more transparent than the one in the 5th ed.). With  $N(x, r)$  the number of  $n \leq x$  not divisible by any of the first  $r$  primes  $p_k$ , then

$$\pi(x) \leq N(x, r) + r$$

(a prime  $p \leq x$  is either one of the first  $r$  or not divisible by any of the first  $r$ ). By Inclusion-Exclusion (Problems 4 Q2),

$$N(x, r) = [x] - \sum_i [x/p_i] + \sum_{ij} [x/p_i p_j] \dots$$

The number of square brackets is

$$1 + \binom{r}{1} + \binom{r}{2} + \dots = (1+1)^r = 2^r.$$

Replacing each  $[.]$  by  $.$  introduces an error of  $< 1$ , so

$$N(x, r) < x - \sum_i x/p_i + \sum_{ij} x/p_i p_j \dots + 2^r = x \prod_1^r (1 - 1/p_k) + 2^r.$$

Combining,

$$\pi(x) \leq x \prod_1^r (1 - 1/p_k) + 2^r + r : \quad \pi(x)/x \leq \prod_1^r (1 - 1/p_k) + (2^r + r)/x.$$

As the product diverges (Q1),  $\prod_1^r$  can be made arbitrarily small by taking  $r$  large enough. Then letting  $x \rightarrow \infty$  gives  $\pi(x)/x \rightarrow 0$ . //

Q3 (A, Th. 2.15 p.35-6). By contradiction: we assume  $a$  is not multiplicative and deduce that  $a * b$  is not multiplicative. Let  $c := a * b$ . As  $a$  is not multiplicative, there are positive integers  $m, n$  with  $(m, n) = 1$  but  $a(mn) \neq a(m)a(n)$ . Choose the pair  $m$  and  $n$  with  $mn$  as small as possible.

If  $mn = 1$ , then  $a(1) \neq a(1)a(1)$ , so  $a(1) \neq 1$ . As  $b(1) = 1$  ( $b$  multiplicative) and  $c(1) = a(1)b(1)$  ( $c := a * b$  is multiplicative),  $c(1) = a(1)b(1) = a(1) \neq 1$ , this shows that  $c = a * b$  is not multiplicative, a contradiction.

If  $mn > 1$ , then by minimality of  $mn$ ,  $a(m'n') = a(m')a(n')$  for all coprime  $m', n'$  with  $m'n' > mn$ . So (as in II.3 Prop.)

$$\begin{aligned} c(mn) &= \sum_{j|m, k|n, jk < mn} a(jk)b(mn/jk) + a(mn)b(1) \\ &= \sum_{j|m, k|n, jk < mn} a(j)a(k)b(m/j)b(n/k) + a(mn) \\ &= \sum_{j|m} a(j)b(m/j) \sum_{k|n} a(k)b(n/k) - a(m)b(n) + a(mn) \\ &= c(m)c(n) - a(m)b(n) + a(mn). \end{aligned}$$

As  $a(mn) \neq a(m)a(n)$ ,  $c(mn) \neq c(m)c(n)$ , contradicting  $c$  multiplicative. //

Q4. By II.3 L8,  $u * u = d$ . By II.5 L10,  $u * \mu = e$ . Combining,

$$d * \mu = (u * u) * \mu = u * (u * \mu) = u * e = u.$$

The Dirichlet series of  $u$  is  $\zeta$ ; that of  $d$  is  $\zeta^2$  (II.4 L8); that of  $\mu$  is  $1/\zeta$  (II.6 L10). So the corresponding identity is  $\zeta^2 \cdot (1/\zeta) = \zeta$ .

$$\begin{aligned} \text{Q5. } D(x) &:= \sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d|n} 1 = \sum_{md \leq x} 1 = \sum_{d \leq x} \sum_{m \leq x/d} 1 \\ &= \sum_{d \leq x} [x/d] = \sum_{d \leq x} (x/d + O(1)) = x(\log x + O(1)) + O(x) \\ &= x \log x + O(x). \end{aligned}$$

NHB