m3pm16soln7(14).tex

M3PM16/M4PM16 SOLUTIONS 7. 7.3.2014

Q1 Prime divisor functions. If $n = p_1^{r_1} \dots p_k^{r_k}, m = q_1^{s_1} \dots q_\ell^{s_\ell}$,

$$\Omega(mn) = r_1 + \ldots + r_k + s_1 + \ldots + s_\ell = \Omega(m)\Omega(n)$$

for all m, n, while $\omega(mn) = k + \ell = \omega(m) + \omega(n)$ if (m, n) = 1. So $(-)^{\Omega}(mn) = (-)^{\Omega(m)} \cdot (-)^{\Omega(n)}$ for all $m, n, (-)^{\omega(mn)} = (-)^{\omega(m)} \cdot (-)^{\omega(n)}$ for m, n coprime. So $(-)^{\Omega}$ is completely multiplicative, $(-)^{\omega}$ is multiplicative.

Q2 The Liouville function λ . (i) For n = p prime, $\Omega(p) = -1$, so $\lambda(p) = -1$. By Q1, λ is completely multiplicative. So by II.7 L11 (Squares and square-free numbers: J p.67): take $a(n) := \lambda(n)/n^s$. Then as a is completely multiplicative as λ is,

$$\frac{\sum \lambda(n)/n^s}{\sum \lambda^2(n)/n^{2s}} = \prod_p \left(1 + \frac{\lambda(p)}{p^s}\right) = \prod_p \left(1 - \frac{1}{p^s}\right) = 1/\zeta(s),$$

by the Euler product (II.5 L9). As $\lambda^2 \equiv 1$, $\sum \lambda^2(n)/n^{2s} = \sum 1/n^{2s} = \zeta(2s)$:

$$\sum_{n=1}^{\infty} \lambda(s)/n^s = \zeta(2s)/\zeta(s).$$

(ii) By II.7 L9 again, $\sum_{n=1}^{\infty} |\mu|(s)/n^s = \zeta(s)/\zeta(2s)$. The product of these two Dirichlet series is 1. So in convolution language,

$$\lambda * |\mu| = e.$$

Q3. With $\nu(n) := \mu(d)$ if $n = d^2$ is a square, 0 otherwise:

Given n, extract from n the product of its prime factors raised to their highest even power. This is a square, m^2 say, and then $n = m^2 q$, with q a product of distinct primes (those occurring in n with odd multiplicity). So $|\mu|(q) = 1$. So $|\mu(n)| = e(m)$, since if m = 1, n = q, so $|\mu(n)| = |\mu(q)| = 1$, while if m > 1, n has a square factor, so both sides are 0. But $\mu * u = e$, so

$$\begin{aligned} |\mu(n)| &= e(m) = \sum_{d|m} \mu(d) &= \sum_{d^2|n} \mu(d) \qquad (d|m \text{ iff } d^2|n) \\ &= \sum_{d^2|n} \nu(d^2) \qquad (\text{definition of } \nu) \\ &= \sum_{d|n} \nu(d) \qquad (\nu(d) = 0 \text{ if } d \text{ is not a square}). \end{aligned}$$

This says that $|\mu| = \nu * u$. Or: $|\mu| = \mu^2 = I_Q$ has Dirichlet series $\zeta(s)/\zeta(2s)$ (II.7 L11); u has Dirichlet series $\zeta(s)$; ν has Dirichlet series

$$\sum_{n} \nu(n)/n^{s} = \sum_{d} \mu(d)/d^{2s} = 1/\zeta(2s),$$

as $\nu(n) := \mu(d)$ if $n = d^2$, 0 otherwise.

Q4 (HW §18.6, Th. 333). (i) For integers $n \leq y^2$, let S(d) be the set of n with biggest square factor d^2 . So S(1) is the set of square-free $n \leq y^2$. Then

$$|S(d)| = Q(y^2/d^2)$$
:

for $n \in S(d)$ iff $n = d^2m \leq y^2$ with *m* square-free, i.e. $m \leq y^2/d^2$ is square-free, and there are $Q(y^2/d^2)$ such *m*, so this many $n = d^2m$.

As Q(x) = 0 for x < 1, $Q(y^2/d^2) = 0$ for d > y, i.e. S(d) is empty for d > y. So as the S(d) form a partition of $\{n \le y^2\}$,

$$[y^2] = \sum_{d \le y} Q(y^2/d^2) : \qquad [m] = \sum_{m \le \sqrt{x}} Q(x/m^2).$$

(ii) By Möbius inversion,

$$Q(y^2) = \sum_{d \le y} \mu(d) [y^2/d^2]: \qquad Q(x) = \sum_{m \le \sqrt{x}} \mu(m) [x/m^2].$$

(iii) Write $[.] = . - \{.\} = . + O(1)$:

$$Q(y^2) = \sum_{d \le y} \mu(d)(y^2/d^2 + O(1)) = \sum_{d=1}^{\infty} \mu(d)/d^2 + O(y^2 \sum_{d > y} 1/d^2) + O(y),$$

as $|\mu| \leq 1$. For large y, $\sum_{d>y} 1/d^2 \sim \int_y^\infty dx/x^2 = 1/y$. So both error terms are O(y), and we can combine them. As μ has Dirichlet series $1/\zeta$, $\sum_{d=1}^\infty 1/d^2 = 1/\zeta(2)$. But $\zeta(2) = \pi^2/6$ (Basel problem: Problems 4 Q3), so

$$Q(y^2) = \frac{6}{\pi^2}y^2 + O(y).$$

Replacing y by \sqrt{x} gives the result. //

NHB