m3pm16const.tex

CONSTANTS LEMMA

LEMMA (CONSTANTS LEMMA: HW Th. 428, 351-3). In Mertens' Second Theorem,

$$C_1 = \gamma + \sum_p \Big(\log(1 - \frac{1}{p}) + \frac{1}{p}\Big), = \gamma + \Sigma, \quad \text{say.}$$

Proof. First, the sum defining Σ converges. For, $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. So

$$0 < -\log(1 - 1/p) - 1/p = \frac{1}{2p^2} + \frac{1}{3p^3} + \ldots < \frac{1}{2p^2} + \frac{1}{2p^3} + \ldots,$$

summing the GP. Also $\sum_p 1/(p(p-1)) < \sum_n 1/(n(n-1)) < \infty$. So by the Comparison Test,

$$\sum_p \Big(\log(1 - \frac{1}{p}) + \frac{1}{p}\Big) \text{ converges.}$$

Note that $\sum_p 1/p$ diverges (Euler, II.4), which with the above shows that $\sum_p \{-\log(1 - 1/p)\}$ diverges also, i.e. $\prod_p (1 - 1/p)$ diverges to 0 (I.5).

For $\delta > 0$, as above,

$$0 < -\log(1 - \frac{1}{p^{1+\delta}}) - \frac{1}{p^{1+\delta}} < \frac{1}{2p^{1+\delta}(p^{1+\delta} - 1)} \le \frac{1}{2p(p-1)},$$

uniformly in $\delta \ge 0$. So

$$F(\delta) := \sum_p \Big(\log(1 - \frac{1}{p^{1+\delta}}) + \frac{1}{p^{1+\delta}}\Big)$$

converges, uniformly in $\delta \ge 0$. So $\qquad F(\delta) \to F(0) \qquad (\delta \downarrow 0). \qquad (a)$

Take $\delta > 0$. By the Euler product (II.4),

$$F(\delta) = g(\delta) - \log\zeta(1+\delta), \qquad g(\delta) := \sum_p 1/p^{1+\delta}.$$

With $E(x) (= O(1/\log x))$ the error term in Mertens' Second Theorem, $c_p := 1/p$ if p is prime, 0 otherwise, Mertens' Second Theorem is

$$C(x) := \sum_{n \le x} c_n = \sum_{p \le x} 1/p = \log\log x + C_1 + E(x).$$

By Abel summation with $f(x) := x^{-\delta}$,

$$\sum_{p\le x} p^{-1-\delta} = x^{-\delta}C(x) + \sigma\int_2^x t^{-1-\delta}C(t)dt.$$

Letting $x\to\infty$, this becomes

$$g(\delta) = \delta\int_2^\infty t^{-1-\delta}C(t)dt = \delta\int_2^\infty t^{-1-\delta}(\log\log t + C_1)dt + \delta\int_2^\infty t^{-1-\delta}E(t)dt. \qquad (*)$$

Now $\delta\int_1^\infty t^{-1-\delta}\log\log t dt = \int_1^\infty e^{-u}\log(u/\delta)du$ $(t = e^{u/\delta}) = -\gamma - \log\delta$ (as $\int_0^\infty e^{-u}\log u du = -\gamma$ by I.8 and $\int_0^\infty e^{-u}du = 1$), and $\delta\int_1^\infty t^{-1-\delta}dt = 1$. Substituting in $(*)$,

$$g(\delta) + \log\delta - C_1 + \gamma = \delta\int_2^\infty t^{-1-\delta}E(t)dt - \delta\int_1^2 t^{-1-\delta}(\log\log t + C_1)dt. \qquad (**)$$

For the first term on RHS in $(**)$: write A for a generic constant (rather than using O-notation as we usually do), and recall $E(t) = O(1/\log t)$. Then with $T := \exp(1/\sqrt{\delta})$,

$$|\delta\int_2^\infty \frac{E(t)}{t^{1+\delta}}dt| < A\delta\int_2^T \frac{dt}{t} + \frac{A\delta}{\log T}\int_T^\infty \frac{dt}{t^{1+\delta}} < A\delta\log T + A/\log T < A\sqrt{\delta}\to 0 \quad (\delta\to 0).$$

For the second term on RHS of $(**)$:

$$|\int_1^2 t^{-1-\delta}(\log\log t + C_1)dt| < \int_1^2 t^{-1}(|\log\log t| + |C_1|)dt = A$$

(although $\log\log t$ is unbounded at 1, the integral converges – check). Combining, $(**)$ gives

$$g(\delta) + \log\delta \to C_1 - \gamma \qquad (\delta\to 0).$$

But because ζ has a simple pole at 1 of residue 1,

$$\zeta(s) = 1/(s-1) + O(1), \qquad \log\zeta(s) = \log(1/(s-1)) + O(s-1) \qquad (s\to 1):$$

$$\log\zeta(1+\delta) + \log\delta \to 0 \qquad (\delta\to 0).$$

So

$$F(\delta)\to C_1 - \gamma \qquad (\delta\to 0) \qquad (b)$$

from the definition of $F(\delta)$, $g(\delta)$ above. By (a) and (b),

$$C_1 = \gamma + F(0) = \gamma + \sum_p \Big(\log(1-\frac{1}{p}) + \frac{1}{p}\Big). \qquad //$$