m3p16cwsoln15.tex

M4PM16/M5P16: SOLUTIONS TO ASSESSED COURSEWORK, 5.3.2015

$$li(x) := \int_2^x \frac{du}{\log u} \sim x/\log x \quad \text{gives} \quad li^{-1}(x) \sim x \log x,$$

as in going from $\pi(x) \sim x/\log x$ to $p_n \sim n \log n$ (Problems 1 Q4).

Next, we need an estimate for $(li^{-1})'$. Write f for li, g for li^{-1}:

$$f(g(x)) = x : \qquad f'(g(x)g'(x) = 1 : \qquad g'(x) = 1/f'(g(x)).$$

Here $f(x) = li(x) = \int_2^x du/\log u$, so $f'(x) = 1/\log x$, $1/f'(x) = \log x$. So

$$g'(x) = \log(g(x)) = \log(li^{-1}(x)),$$

and as $li^{-1}(x) \sim x \log x$,

$$(li^{-1})'(x) = g'(x) = \log((li^{-1})'(x) \sim \log(x \log x) = \log x + \log\log x \sim \log x.$$

By PNT-R, $\pi(x) = li(x) + O(xe^{-c\sqrt{\log x}})$; as in Prob. 1 Q4, $\pi(p_n) = n$, giving

$$n = li(p_n) + O(p_n e^{-c\sqrt{\log p_n}}) : \quad p_n = li^{-1}(n + O(p_n e^{-c\sqrt{\log p_n}})).$$

The error term simplifies: $p_n \sim n \log n$, and $e^{-c\sqrt{\log p_n}} \sim e^{-c\sqrt{\log n + \log\log n}} \sim e^{-c\sqrt{\log n}}$, as $e^{-c\sqrt{\log x}}$ is *slowly varying (SV)*, like $\log x$ (though much smaller!): $e^{-\sqrt{\lambda x}}/e^{-\sqrt{x}} \to 1$, as you can check. So

$$p_n = li^{-1}(n + O(n \log n.e^{-c\sqrt{n}})) = g(n + O(n \log n.e^{-c\sqrt{n}})).$$

By the Mean Value Theorem, for some $\theta \in (0,1)$,

$$p_n = g(n) + O(n \log n.e^{-c\sqrt{\log n}}).g'(n + \theta.n \log n.e^{-c\sqrt{\log n}}),$$

and as $g'(x) \sim \log x$ is SV as above, $g'(n+\theta.n \log n.e^{-c\sqrt{\log n}}) \sim g'(n) \sim \log n$:

$$p_n = li^{-1}(n) + O(n \log n. \log n.e^{-c\sqrt{\log n}}) = li^{-1}(n) + O(n \log^2 n.e^{-c\sqrt{\log n}}). \quad //$$

Note. 1. I haven't seen this in a modern text. I found it in an old one: Harald Bohr & Harald Cramér, Die neuere Entwicklung der analytischen Zahlentheorie [The more recent development of analytic number theory]. II.C.8, *Encykl. Math. Wiss.* II 3 (1923), §31.
This is reprinted in two sources, both of which I have:
H. Bohr, *Collected mathematical works*, Volume III, Danish Math. Soc., 1952, 722-849;
H. Cramér, *Collected Works*, Volume I, Springer, 1994, 289-416.
2. By *Cipolla's formula* (M. Cipolla, 1902),

$$p_n \sim li^{-1}(n) = n\Big(\log n+\log\log n-1+\frac{\log\log n-2}{\log n}+\frac{\log^2(n)-6\log\log n+11}{2\log^n}+\ldots\Big).$$

In fact (Rosser's theorem: J. B. Rosser (1938), P. Dusart (1999)) this asymptotic expansion is an inequality if truncated after the first three terms:

$$p_n > n(\log n + \log\log n - 1).$$

So Cipolla's formula contains a series of estimates for p_n, all vastly inferior to the Bohr-Cramér result above – which contains them all, *plus* a vastly superior error term.
3. For background on asymptotic expansions of such inverse functions, see B. Salvy, Fast computation of some asymptotic functional inverses. *J. Symbolic Comput.* **17** (1994), 227-236.
Computer algebra is widely used nowadays to do the sort of calculation above – imagine how laborious it would be to do this sort of thing by hand! In some sense, it would also be pointless: there are infinitely many approximations here, all much worse than the li^{-1} result above. But it is good to see how this compares with what we knew already.
4. The 'last word' on this as of now seems to be
J. Arias de Reyna and J. Toulisse, The nth prime, asymptotically. *J. Th. Nombres, Bordeaux* **25** no. 3 (2013), 521-555 (MR3179675).

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