M3PM16/M4PM16 EXAMINATION 2012

Four questions, 20 marks per question

Q1. With $\sum_p$ denoting a sum over all primes p, show that
(i) $\sum_p 1/p$ diverges;
(ii) $\sum_p 1/(p \log p)$ converges.

Q2. Show that
(i) there are arbitrarily long gaps between primes;
(ii) there are infinitely many primes of the form $4n + 3$;
(iii) there are infinitely many primes of the form $6n + 5$.
(iv) State without proof the Bertrand postulate.

Q3. (i) Show how to continue the zeta function $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ analytically from $Re\ s > 1$ to $Re\ s > 0$, and show that it has a simple pole at $s = 1$ of residue 1.
(ii) Show that $\zeta(2) = \pi^2/6$.

Q4. Write $d(n)$ for the number of divisors of a natural number n and set $D(x) := \sum_{n \leq x} d(n)$ for each real number x. Show that
(i) $\sum_{n=1}^{\infty} d(n)/n^s = \zeta(s)^2$ $(Re\ s > 1)$;
(ii) $D(x) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$ $(x \to \infty)$,
where γ denotes Euler's constant.

N. H. Bingham