M3PM16/M4PM16 EXAMINATION 2013

Q1. (i) Assuming the Prime Number Theorem (PNT) in the form $\pi(x) \sim x/\log x$, show that, with p_n the nth prime,

$$p_n \sim n \log n.$$

(ii) With $d_n := p_{n+1} - p_n$, show (by (i) and Abel summation, or otherwise) that

$$\sum_{1<n\leq x} \frac{d_n}{\log n} \sim x \qquad (x \to \infty).$$

(iii) Hence or otherwise show that

$$\liminf_{n\to\infty}(d_n/\log n) \leq 1 \leq \limsup_{n\to\infty}(d_n/\log n).$$

Q2. Given Mertens' first theorem in the form

$$\sum_{p\leq x} \log p/p = \log x + O(1),$$

prove Mertens' second theorem:

$$\sum_{p\leq x} 1/p = \log\log x + C + O(1/\log x),$$

for some constant C.

Q3. Define the *Möbius function* μ and the *von Mangoldt function* Λ. Show that
(i) $\sum_{d|n} \mu(d) = 1$ if $n = 1$, 0 if $n > 1$;
(ii) If $n > 1$ has k distinct prime factors, $\sum_{d|n} |\mu(d)| = 2^k$;
(iii) $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d$.
(iv) By considering

$$-\frac{d}{ds}\Big(\frac{1}{\zeta(s)}\Big) = \frac{\zeta'(s)}{\zeta(s)^2} = -\frac{1}{\zeta(s)}\Big(-\frac{\zeta(s)'}{\zeta(s)}\Big),$$

or otherwise, show that

$$-\mu(n)\log n = \sum_{d|n} \mu(n/d)\Lambda(d).$$

(v) Hence or otherwise, show that

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

Q4. Show that
(i) $3 + 4\cos\theta + \cos 2\theta \geq 0$ for all real θ;
(ii) if $f(s) := \sum_1^\infty a_n/n^s$, where $a_n \geq 0$ for all n, is convergent for $\sigma > \sigma_0$, then

$$3f(\sigma) + 4Re\ f(\sigma + it) + Re\ f(\sigma + 2it) \geq 0 \qquad (\sigma > \sigma_0).$$

Hence or otherwise show that $\zeta(.)$ is non-vanishing on the 1-line $\sigma = Re\ s = 1$.

Describe briefly, without proof, why this result is important.

N. H. Bingham