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M3PM16/M4PM16 EXAMINATION SOLUTIONS 2014

Q1 (HW §§5.5, 16.1,2, J, 68-9, A, §§2.3 - 2.5). (i) Using $|.|$ for cardinality, we partition the set $S := \{1, 2, \ldots, n\}$ as a disjoint union of the sets $A(d)$ containing those elements k of S whose gcd with n is d. So $\sum_1^n |A(d)| = n$. But $(k, n) = d$ iff k/d and n/d are coprime, and $0 < k \leq n$ iff $0 < k/d \leq n/d$. So if $q := k/d$, there is a one-one correspondence $k \leftrightarrow q = k/d$ between the elements of $A(d)$ and the integers q with $0 < q \leq n/d$ with q and n/d coprime. The number of such q is $\phi(n/d)$ (definition of ϕ). So

$$\sum_{d|n} \phi(n/d) = n :$$

$$\sum_{d|n} \phi(d) = n : \qquad I = \phi * u. \qquad [\mathbf{6}]$$

(ii) Since μ and u are convolution inverses, this gives

$$I * \mu = \phi * u * \mu = \phi : \qquad \phi(n) = \sum_{d|n} \mu(d) I(n/d) = \sum_{d|n} \mu(d).n/d. \qquad [\mathbf{3}]$$

(iii) Since μ and I are multiplicative, so is $\phi = \mu * I$. **[2]**
(iv) Taking Dirichlet series, as $\mu(n)$, $I(n) = n$ have Dirichlet series $1/\zeta(s) = \sum_1^\infty \mu(n)/n^s$, $\zeta(s-1) = \sum_1^\infty n/n^s = \sum_1^\infty 1/n^{s-1}$, this gives the Dirichlet series of ϕ as

$$\sum_1^\infty \phi(n)/n^s = \zeta(s-1)/\zeta(s). \qquad [\mathbf{4}]$$

(v) Being multiplicative, ϕ is determined by its values on prime powers p^c, as prime powers of distinct primes are coprime. There are $p^c - 1$ positive integers $< p^c$, of which the multiples of p are $p, 2p, \ldots, p^c - p$ (so $p^{c-1} - 1$ of these), and the rest are coprime to p^c. So

$$\phi(p^c) = (p^c - 1) - (p^{c-1} - 1) = p^c - p^{c-1} = p^c(1 - \frac{1}{p}).$$

So if $n = \prod p^c$ is the prime-power factorisation of n (FTA), (ii) gives

$$\phi(n) = \prod \phi(p^c) = \prod p^c \prod (1 - \frac{1}{p}) = n \prod_{p|n} (1 - \frac{1}{p}). \qquad // \ [\mathbf{5}]$$

[Seen, Problems]

Q2. (i) Mertens' Second Theorem: $\sum_{p\leq x} 1/p = \log\log x + C_1 + O(1/\log x)$ for some constant C_1. **[3]**
(ii) Mertens' Second Theorem for prime powers:

$$\sum_{p^n\leq x} 1/p^n = \log\log x + C_2 + O(1/\log x), \qquad C_2 := C_1 + S, \qquad S := \sum_p \frac{1}{p(p-1)}.$$

Proof. Write $q := p^n$ for a generic prime power, and for primes p with $p^2 \leq x$, let r_p be the largest 'relevant power' (largest r with $p^r \leq x$). Then

$$\Delta := \sum_{q\leq x} 1/q - \sum_{p\leq x} 1/p = \sum_{p\leq\sqrt{x}} \sum_{r=2}^{r_p} 1/p^r.$$

But $\sum_2^\infty 1/p_r = 1/(p(p-1))$, summing the GP, so

$$\Delta \leq \sum_p \frac{1}{p(p-1)} = S$$

(above). With

$$S_0 := \sum_{p\leq\sqrt{x}} \frac{1}{p(p-1)},$$

$$\begin{aligned} S - S_0 = \sum_{p>\sqrt{x}} \quad &< \quad \sum_{n>\sqrt{x}} \frac{1}{n(n-1)} \\ &= \quad \frac{1}{\sqrt{x}} \qquad \left(\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \text{ sum telescopes}\right) \\ &\leq 2/\sqrt{x}. \end{aligned}$$ **[10]**

As $p^{r_p+1} \geq x$:

$$\sum_{r>r_p} \frac{1}{p^r} < \frac{1}{x}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \frac{1}{x(1-1/p)} \leq 2/\sqrt{x} \qquad (p \geq 2).$$

So (as $\pi(x) := \sum_{p\leq x} 1 \leq \sum_{n\leq x} 1 \leq x$)

$$S_0 - \Delta = \sum_{p\leq\sqrt{x}} \sum_{r>r_p} 1/p^r < \pi(\sqrt{x}).2/x \leq 2/\sqrt{x}.$$

Combining, $S - \Delta \leq 4/\sqrt{x} = O(1/\log x)$. **[4]**
So the difference Δ in the sums here and in Mertens' Second Theorem is $S + O(1/\log x)$, and the result follows from Mertens' Second Theorem. // **[3]**
[Seen – Mock Exam 2012; similar to Mertens' Second Theorem, lectures.]

Q3. (i) For ζ, $\sigma_c = \sigma_a = 1$; for η, $\sigma_a = 0$, $\sigma_a = 1$. **[2]**
(ii) $\eta(\sigma) := \sum_{n=1}^{\infty}(-)^{n-1}/n^{\sigma} = \sum_{odd} 1/n^{\sigma} - \sum_{even} 1/n^{\sigma} = \sum_o - \sum_e$, say. Now $\sum_e = \sum_1^{\infty} 1/(2n)^{\sigma} = 2^{-\sigma}\sum_1^{\infty} 1/n^{\sigma} = 2^{-\sigma}\zeta(\sigma)$. So

$$\eta(\sigma) = \sum_o - \sum_e = \sum_o -2^{-\sigma}\zeta(\sigma), \qquad \zeta(\sigma) = \sum_o + \sum_e = \sum_o +2^{-\sigma}\zeta(\sigma).$$

Subtract: $\eta(\sigma) - \zeta(\sigma) = -2.2^{-\sigma}\zeta(\sigma) = -2^{1-\sigma}\zeta(\sigma)$:

$$\eta(\sigma) = (1-2^{1-\sigma})\zeta(\sigma): \quad \zeta(\sigma) = (1-2^{1-\sigma})^{-1}.\sum_1^{\infty}(-)^{n-1}/n^{\sigma}. \qquad (*)$$

Note that the series on RHS converges in $\sigma > 0$, by the Alternating Series Test. Now let s be complex, and define

$$\zeta(s) := \eta(s)/(1-2^{1-s}) = (\sum_{n=1}^{\infty}(-)^{n-1}/n^{s})/(1-e^{(1-s)\log 2}). \qquad (**)$$

As $\eta(s)$ has $\sigma_c = 0$, the first factor on RHS is holomorphic in $\sigma > 0$ (the second is holomorphic as $2^{-s} = e^{-s\log 2}$ is). So we may use $(*)$ to *continue* $\zeta(s)$ *analytically* from $\sigma > 1$ to $\sigma > 0$. **[6]**
(iii) $\sum_{odd} > \sum_{even}$ (compare corresponding terms). So $\eta(\sigma)$ has no zeros in $\sigma \in (0,1)$. So $\zeta(\sigma)$ has none either: any zeros of ζ in the critical strip $0 < \sigma < 1$ are non-real. **[2]**
(iv)Near $s = 1$, $1 - 2^{1-s} = 1 - e^{(1-s)\log 2} = (s-1)\log 2 + O((s-1)^2)$. As $\eta(1) = \log 2$, given, $\log 2$ cancels in $(**)$ to give $\zeta(s) \sim 1/(s-1)$: $\zeta(s)$ is holomorphic in $Res > 0$ except for a simple pole at 1 of residue 1. // **[3]**
(v) $\zeta(0) = -\frac{1}{2}$; $\zeta(-2n) = 0$ $(n = 1, 2, \ldots)$.
Proof. Γ has a simple pole at 0 of residue 1; ζ has a simple pole at 1 of residue 1. So near $s = 0$, (FE) and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ give

$$\frac{2}{s}.\zeta(s) \sim \frac{1}{\sqrt{\pi}}.\Gamma(\frac{1}{2}).(-\frac{1}{s}) = -\frac{1}{s}: \qquad \zeta(0) = -\frac{1}{2}. \qquad [2]$$

The RHS of (FE) is holomorphic at $s = 2n$. The LHS contains a (simple) pole from $\Gamma(-\frac{1}{2}s)$, so this must be cancelled by a (simple) zero of ζ: $\zeta(-2n) = 0$. (The zeros of ζ at $-2n$ are called the *trivial zeros*.) **[2]**
(vi) All zeros of ζ other than the trivial zeros lie in the critical strip $0 < \sigma < 1$.
Proof. The RHS of (FE) is holomorphic in $\sigma > 1$, and non-zero there (there are no zeros in $\sigma > 1$ by the Euler product, and none on the 1-line, given). So the LHS of (FE) is holomorphic and non-zero in $\sigma < 0$. But the only poles of Γ are $0, -1, \ldots, -n, \ldots$. So the only zeros of ζ in $\sigma < 0$ are the trivial zeros that cancel these. The remaining zeros are in the critical strip. **[3]**
[Seen in lectures, except for (iii), unseen.]

Q4. (i). With $\nu(n) := \mu(d)$ if $n = d^2$ is a square, 0 otherwise:
Given n, extract from n the product of its prime factors raised to their highest *even* power. This is a square, m^2 say, and then $n = m^2 q$, with q a product of distinct primes (those occurring in n with *odd* multiplicity). So $|\mu|(q) = 1$. So $|\mu(n)| = e(m)$, since if $m = 1$, $n = q$, so $|\mu(n)| = |\mu(q)| = 1$, while if $m > 1$, n has a square factor, so both sides are 0. But $\mu * u = e$, so

$$\begin{aligned} |\mu(n)| = e(m) = \sum_{d|m} \mu(d) &= \sum_{d^2|n} \mu(d) \qquad (d|m \text{ iff } d^2|n) \\ &= \sum_{d^2|n} \nu(d^2) \qquad (\text{definition of } \nu) \\ &= \sum_{d|n} \nu(d) \qquad (\nu(d) = 0 \text{ if } d \text{ is not a square}). \end{aligned}$$

So $|\mu| = \nu * u$. Or: $|\mu| = \mu^2 = I_Q$ has Dirichlet series $\zeta(s)/\zeta(2s)$ (II.7 L11); u has Dirichlet series $\zeta(s)$; ν has Dirichlet series $1/\zeta(2s)$ (check). **[6]**
(ii) For integers $n \leq y^2$, let $S(d)$ be the set of n with biggest square factor d^2. So $S(1)$ is the set of *square-free* $n \leq y^2$. Then

$$|S(d)| = Q(y^2/d^2) :$$

for $n \in S(d)$ iff $n = d^2 m \leq y^2$ with m square-free, i.e. $m \leq y^2/d^2$ is square-free, and there are $Q(y^2/d^2)$ such m, so this many $n = d^2 m$.

As $Q(x) = 0$ for $x < 1$, $Q(y^2/d^2) = 0$ for $d > y$, i.e. $S(d)$ is empty for $d > y$. So as the $S(d)$ form a partition of $\{n \leq y^2\}$,

$$[y^2] = \sum_{d \leq y} Q(y^2/d^2) : \qquad [x] = \sum_{m \leq \sqrt{x}} Q(x/m^2). \qquad \textbf{[6]}$$

(iii) By Möbius inversion of (i), $Q(x) = \sum_{m \leq \sqrt{x}} \mu(m)[x/m^2]$. **[2]**
(iv) Write $[.] = . - \{.\} = . + O(1)$:

$$Q(y^2) = \sum_{d \leq y} \mu(d)(y^2/d^2 + O(1)) = y^2 \sum_{d=1}^{\infty} \mu(d)/d^2 + O(y^2 \sum_{d > y} 1/d^2) + O(y),$$

as $|\mu| \leq 1$. For large y, $\sum_{d>y} 1/d^2 \sim \int_y^\infty dx/x^2 = 1/y$. So both error terms are $O(y)$, and we can combine them. As μ has Dirichlet series $1/\zeta$, $\sum_{d=1}^\infty 1/d^2 = 1/\zeta(2)$. But $\zeta(2) = \pi^2/6$ (Euler: Basel problem), so

$$Q(y^2) = \frac{6}{\pi^2} y^2 + O(y) : \qquad Q(x) = \frac{6}{\pi^2} x + O(\sqrt{x}). \qquad // \textbf{[6]}$$

[Seen – Problems] N. H. Bingham