m3pm16l12.tex

Lecture 12. 5.2.2015

$\log \zeta(s)$

Theorem. Write $c_n := 1/m$ if $n = p^m$ is a prime power, 0 otherwise. Then

$$H(s) := \sum_1^\infty c_n/n^s = \log \zeta(s) \qquad (Re\ s > 1).$$

Proof. $\zeta(s) = \prod 1/(1-p^{-s})$, so $\log(\zeta(s)) = -\sum_p \log(1-p^{-s})$. As $-\log(1-z) = \sum_1^\infty z^m/m$ for $|z| < 1$,

$$\log \zeta(s) = \sum_p \sum_1^\infty \frac{1}{m}.1/p^{ms} = \sum_{m=1}^\infty \frac{1}{m} \sum_{n=p^m} 1/n^s = \sum_{n=1}^\infty c_n/n^s = H(s),$$

in the half-plane of absolute convergence $Re\ s > 1$, where changing of order of summation is justified. The function $\log \zeta(s)$ is holomorphic on $\sigma > 1$ as $\zeta(s)$ is holomorphic and nonzero there. //

The Von Mangoldt Function $\Lambda(n)$, 1899.

$\Lambda(n) := \log(p)$ if $n = p^m$ is a prime power, 0 otherwise. Differentiating the result above gives ($Dn^{-s} = De^{-s\log n} = -\log n.e^{-s\log n} = -\log n.n^{-s}$)

$$-\zeta'(s)/\zeta(s) = \sum_1^\infty c_n \log n/n^s.$$

But

$$c_n \log n = \frac{1}{m} \cdot m \log p = \log p \text{ if } n = p^m, 0 \text{ otherwise}: \quad c_n \log n = \Lambda(n).$$

So (this is one of the most important formulae of the course – *please learn*!)

$$-\zeta'(s)/\zeta(s) = \sum\nolimits_1^\infty \Lambda(n)/n^s \qquad (Re\ s > 1). \qquad (*)$$

Corollary. With $\ell(n) := \log(n)$, $\Lambda * \mathbf{1} = \ell, \ell * \mu = \Lambda$.

Proof. $\Lambda(1) = \ell(1) = 0$. For $n > 1, n = p_1^{r_1}...p_k^{r_k}$, say. Then $(\Lambda * \mathbf{1})(n) = \sum_{i|n} \Lambda(i)$. The divisors i of n are $i = p_1^{s_1}...p_1^{s_k}, 0 \leq s_j \leq r_j$. Those with $\Lambda(i) \neq 0$ are only those with $i = p_j^{s_j}$, each of which has $\Lambda(i) = \log p_j$. Then

$$\sum_{i|n} \Lambda(i) = \sum_{j=1}^k r_j \log p_j = \log \prod_j p_j^{r_j} = \log n = \ell(n)$$

($1 \leq s_j \leq r_j$ here: $s_j = 0$ gives $n = 1$, dealt with already). So $\Lambda * \mathbf{1} = \ell$, and then $\ell * \mu = \Lambda$ follows by Möbius inversion. //

Write

$$\psi(x) := \sum_{n \leq x} \Lambda(n), \ = \sum_{p^m \leq x} \log p$$

(so $\psi(1) = 0$ and $\psi(x) = O(x \log x)$ as $\Lambda(n) = O(\log n)$). As the highest power m with $p^m \leq x$ is $m = \left[\frac{\log x}{\log p}\right]$: $\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p$. By Abel summation (I.3 – integration by parts for Stieltjes integrals), $(*)$ is

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum\nolimits_1^\infty \Lambda(n)/n^s = \int_1^\infty d\psi(x)/x^s = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \quad (Re\ s > 1). \tag{$**$}$$

We shall see that PNT ($\pi(x) \sim li(x) = x/\log x$) is equivalent to

$$\psi(x) \sim x,$$

and this is how we prove it (Ch. III; remainder version in Ch. IV). The key formula for us is $(*)$, and we shall need it on the line $\sigma = Re\ s = 1$, the *1-line*. For good behaviour there, one needs *non-vanishing of zeta on the 1-line*: $\zeta(1 + it) \neq 0$ for $t \neq 0$ (III.4).

7. Mertens' Theorems

Theorem (HW Th. 424). For $x > 1$,

$$\sum_{n \leq x} \Lambda(n)/n = \log x + r(x), \text{ with } |r(\cdot)| \leq 2.$$

Proof. By II.5, $\left|\sum_1^N \mu(n)/n\right| \leq 1$. Write $S(x) := \sum_{n \leq x} \log n$. As $\Lambda * \mathbf{1} = \ell$,

$$S(x) = \sum_{n \leq x} (\Lambda * \mathbf{1})(n) = \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] \qquad \text{(with [.] the integer part)}$$

$$= x \sum_{n \leq x} \frac{\Lambda(n)}{n} - a(x), \qquad \text{where} \qquad a(x) := \sum_{n \leq x} -\Lambda(n)\{x/n\},$$

with $\{.\}$ the fractional part. Then

$$0 \leq a(x) = \sum_{n \leq x} \Lambda(u)\{x/n\} \leq \sum_{n \leq x} \Lambda(u) = \psi(x) \leq 2x,$$

by Chebyshev's Upper Estimate (III.2 below).