

Now use the Integral Test argument (I.4) to estimate  $S(x)$ . As

$$\int_1^x \log t dt = x \log x - x + 1$$

(integrate by parts), this gives

$$S(x) = x \log x - x + b(x), \quad |b(x)| \leq \log x + 1.$$

Now  $\log x + 1 \leq x$  for  $x > 1$  (integrate  $1/x \leq 1$  over  $[1, x]$ ). So  $|b(x)| \leq x$ . So

$$x \sum_{n \leq x} \Lambda(n)/n = S(x) + a(x) = x \log x - x + a(x) + b(x).$$

But  $0 \leq a(x) \leq 2x$ ,  $|b(x)| \leq x$ , so  $|a(x) - x + b(x)| \leq 2x$ . //

**Cor.**

$$\int_1^x \frac{\psi(t)}{t^2} dt = \log x + O(1) \quad (x > 1).$$

*Proof.* Integrating by parts (or by Abel summation (I.3)),

$$\sum_{n \leq x} \Lambda(n)/n = \int_{(1,x]} d\psi(x)/x = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2} dt.$$

But  $\psi(x)/x$  is bounded (from Chebyshev's  $\theta$ -function: III.2), so this follows from the Theorem. //

The next result shows that we can neglect the powers of primes in the Theorem (at the cost of losing the bound 2): *powers of primes become sparse*, so this is not too surprising.

**Theorem (Mertens' First Theorem:** F. MERTENS (1840-1927) in 1874; HW Th. 425).

$$\sum_{p \leq x} \log p/p = \log x + O(1) \quad (x > 1) \quad (|O(\cdot)| \leq 4).$$

*Proof.* As  $\Lambda(n) = \log p$  when  $n = p^m$ ,

$$0 \leq \sum_{n \leq x} \Lambda(n)/n - \sum_{p \leq x} \log p/p = \sum_{p^m \leq x} \log p/p - \sum_{p \leq x} \log p/p$$

$$= \sum_{m \geq 2} \sum_{p^m \leq x} \log p/p \leq \sum_{p \leq x} \log p \left( \frac{1}{p^2} + \frac{1}{p^3} + \dots \right)$$

Summing the geometric series, the RHS is

$$\sum_{p \leq x} \frac{\log p}{p(p-1)} \leq \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)} < \infty$$

(convergent, to sum  $\leq 2$  – check), giving the result by the Theorem above. //

**Theorem (Mertens' Second theorem; HW Th. 427).**

$$\sum_{p \leq x} 1/p = \log \log x + C_1 + O(1/\log x) \quad (x \geq 2),$$

for some constant  $C_1$ .

*Proof* (Compare  $\sum_{n \leq x} 1/n = \log x + \gamma + o(1)$ , I.4). Write

$$a(n) := \log n/n \quad (n \text{ prime}), \quad 0 \text{ otherwise}, \quad A(x) := \sum_{n \leq x} a_n$$

(so  $a(1) = 0$ ). By Mertens' First Theorem,

$$A(x) = \log x + r(x), \quad |r(\cdot)| \leq c_0 \quad (x > 1),$$

$$\text{and } \sum_{p \leq x} 1/p = \sum_{n \leq x} \frac{a(n)}{\log n} = \int_{(1,x]} dA(u)/\log u = \int_{[2,x]} dA(u)/\log u.$$

Integrating by parts (or by Abel summation), this gives

$$\sum_{p \leq x} 1/p = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t \log^2 t} dt = 1 + \frac{r(x)}{\log x} + \int_2^x \frac{dt}{t \log t} + I(x),$$

$$I(x) := \int_2^x \frac{r(t)}{t \log^2 t} dt.$$

But

$$\int_2^x \frac{1}{t \log t} dt = \log \log x - \log \log 2, \quad \int_2^{\infty} \frac{dt}{t \log^2 t} < \infty, \quad \text{as } \frac{1}{t \log^2 t} = -\frac{d}{dt} \left( \frac{1}{\log t} \right).$$

So  $I(x) \rightarrow I$ , finite, as  $x \rightarrow \infty$ , and

$$I(x) = I - s(x), \quad |s(x)| \leq c_0 \int_x^{\infty} \frac{dt}{t \log^2 t} = \frac{c_0}{\log x}.$$

This gives the result with  $C_1 := 1 - \log \log 2 + I$ . //