m3pm16l14.tex

Lecture 14. 11.2.2015

Theorem(Merten's Formula, HW Th 929).

$$\prod_{p \le x}(1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x} \qquad (x \to \infty).$$

Proof. Write $\Sigma := \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$, which is convergent. By Merten's Second Theorem and the Constants Lemma (from the Website),

$$\sum_{p \le x} \frac{1}{p} = \log\log x + C_1 + o(1) = \log\log x + \gamma + \Sigma + o(1).$$

Now,

$$\sum_{p \le x}\left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = \Sigma + o(1),$$

from the definition of Σ. Subtracting:

$$\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) = -\log\log x - \gamma + o(1):$$

$$\log\left[\prod_{p \le x}\left(1 - \frac{1}{p}\right)\right] = \log\left[\frac{e^{-\gamma}}{\log x}\right] + o(1):$$

$$\log\left[\prod_{p \le x}(1 - \frac{1}{p})/\frac{e^{-\gamma}}{\log x}\right] \to 0 \qquad (x \to \infty).$$

So [...] $\to 1$, i.e.

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \qquad //$$

Cor.

$$\prod_{p \le x}(1 + \frac{1}{p}) \sim \frac{6}{\pi^2}.e^{\gamma}\log x.$$

Proof. By Euler's solution to the Basel problem (Problems 4 Q3) and the Euler product (II.5 L9),

$$\frac{\pi^2}{6} = \zeta(2) = \sum_1^\infty 1/n^2 = \prod_p 1/(1 - \frac{1}{p^2}): \qquad \prod_{p \le x}(1 - \frac{1}{p})(1 + \frac{1}{p}) \to \frac{6}{\pi^2}.$$

Dividing this by Mertens' formula gives the result. //

9. Dirichlet's Hyperbola Identity (DHI)

Theorem (DHI). If $1 < y < x$,

$$\sum_{n \le x} (a * b)(n) = \sum_{j \le y} a(j)B(x/j) + \sum_{k \le x/y} b(k)A(x/k) - A(y)B(x/y).$$

Proof. LHS $= S := \sum_{jk \le x} a_j b_k$, as in II.3. Write S_1 for the sum of all such terms with $j \le y$, S_2 that of all terms with $k \le x/y$. As in II.3,

$$S_1 = \sum_{jk \le x, j \le y} a_j b_k = \sum_{j \le y} a_j \sum_{k \le x/j} b_k = \sum_{j \le y} a_j B(x/k),$$

the first sum on RHS, and similarly

$$S_2 = \sum_{jk \le x, k \le x/y} a_j b_k = \sum_{k \le x/y} b_k \sum_{j \le x/k} a_j = \sum_{k \le x/y} b_k A(x/k),$$

the second sum on RHS. Now $S_1 + S_2$ counts all terms, but counts twice those with both $j \le y$ and $k \le x/y$. The sum of these terms is $A(y)B(x/y)$. So subtracting this 'corrects the count', and gives the result. //

Theorem. If d_n is the number of divisors of n,

$$\sum_{n \le x} d_n = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Take $a_n = b_n = 1$ (so $(a * b)_n = d_n$, by (i)), $y = \sqrt{x}$: as $A(x) = B(x) = [x]$, Dirichlet's Hyperbola Identity gives

$$\sum_{n \le x} d_n = \sum_{j \le \sqrt{x}} [x/j] + \sum_{k \le \sqrt{x}} [x/k] - [\sqrt{x}][\sqrt{x}] = 2 \sum_{j \le \sqrt{x}} [x/j] - [\sqrt{x}][\sqrt{x}].$$

In each $[.]$ on RHS, write $[.] = . - \{.\}$. Each fractional part $\{.\} \in [0, 1)$, so

$$\sum_{n \le x} d_n = 2 \sum_{j \le \sqrt{x}} x/j + O(\sqrt{x}) - x + O(\sqrt{x}) = 2x \sum_{j \le \sqrt{x}} 1/j - x + O(\sqrt{x}),$$

as $(\sqrt{x} + O(1))^2 = x + O(\sqrt{x})$. But as in L3, I.4,

$$\sum_{j \le \sqrt{x}} 1/j = \log \sqrt{x} + \gamma + O(1/\sqrt{x}) = \frac{1}{2} \log\ x + \gamma + O(1/\sqrt{x}).$$

So

$$\sum_{n \le x} d_n = 2x(\log \sqrt{x} + \gamma + O(1/\sqrt{x})) - x + O(\sqrt{x}) = x \log x + (2\gamma - 1)x + O(\sqrt{x}). \ //$$